

## DIFFERENTIAL EQUATION

Important definition and Results :-

- \* Differential Equation :- An equation containing independent variable, a dependent variable and derivative of dependent variable with respect to independent variable is called the differential equation
- \* Order of differential Equation :- The order of differential equation is the order of the highest order derivative occurring in the equation.
- \* Degree of the differential equations :- The degree of the differential equation is degree of the highest order derivative when differential co-efficient are made free from radicals and fraction.
- \* General Solution :- The solution which contains as many as arbitrary constant as the order as differential equation is called the general solution.
- \* Particular Solution :- The solution obtained by giving particular values to the arbitrary constant in general solution of differential equation is called particular solution.
- \* A differential equation of first order and first degree involves  $x$ ,  $y$  and  $\frac{dy}{dx}$ . So it can be put in any one of the formula  $\frac{dy}{dx} = f(x, y)$  or  $f(x, y)\frac{dy}{dx} = 0$ .
- \* Homogeneous equation :- An equation in  $x$  and  $y$  is said to be homogeneous if it can be put in the form of  $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$  where  $f_1(x, y)$  and  $f_2(x, y)$  are both homogeneous function of same degree in  $x$  and  $y$ .
- \* linear differential equation :- A differential equation is linear if dependent variable ( $y$ ) and its derivative appear only in first degree and general form is  $\frac{dy}{dx} + py = Q$  where  $p$  and  $Q$  are function of  $x$  (or const.)

### (SECTION-A) PRACTICE-EXERCISE

Find the order and degree of the following differential equation

(i)  $y = \frac{xdy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

(ii)  $y''' + y^2 + ey' = 0$

(iii)  $\frac{d^4y}{dx^4} + \sin(y''') = 0$

(iv)  $y''' + (y')^2 + 2y = 0$

(v)  $xy \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx}\right)^2 - y \frac{dy}{dx} = 0$

## SECTION – B

1. Form the differential equation of the family of circle having centre on y-axis and radius 3 units.
2. Form the differential equation of the family of circle touching the y-axis at origin.
3. Form the differential equation of the family of hyperbola having foci on - x-axis.
4. Solve :  $\sin\left(\frac{dy}{dx}\right) = x$
5. Solve :  $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$
6. Solve :  $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$
7. Solve :  $(x^3 + x^2 + x + 1) \frac{dy}{dx} = 2x^2 + x ; y = 1$  when  $x = 0$
8. For differential equation  $xy \frac{dy}{dx} = (x+2)(y+2)$  find the solution curve passing through the point  $(1, -1)$
9. Solve :  $\frac{dy}{dx} - y = \cos x$
10. Solve:  $\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x$ , given that  $y = 0$  when  $x = \pi/2$
11. Solve :  $x \frac{dy}{dx} + 2y = x^2$
12. Solve :  $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$
13. Solve :  $x \frac{dy}{dx} + y - x + xy \cot x = 0$
14. Solve :  $\frac{dy}{dx} - 3y \cot x = \sin 2x ; y = 2$  when  $x = \pi/2$
15. Solve :  $\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x$
16. Solve :  $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$
17. Solve :  $\frac{dy}{dx} = (4x + y + 1)^2$
18. Solve :  $(x - y)(dx + dy) = dx - dy ; y(0) = -1$

## SECTION – C

Solve the following differential equation

1. Solve :  $x^2 dy + y(x+y)dx = 0$  given that  $y=1$  when  $x=1$
2. Solve :  $x^2 y dx - (x^3 + y^3) dy = 0$
3. Solve :  $x dy - y dx = \sqrt{x^2 + y^2} dx$
4. Solve :  $x \frac{dy}{dx} \sin\left(\frac{y}{x}\right) + x - y \sin\left(\frac{y}{x}\right) = 0$  ;  $y(1) = \pi/2$
5. Solve :  $x \frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right)$
6. Show that  $xy = ae^x + be^{-x} + x^2$  is a solution of differential equation  
$$x \frac{d^2y}{dx^2} + \frac{2dy}{dx} - xy + x^2 - 2 = 0$$
7.  $2x^2 \frac{dy}{dx} - 2xy + y^2 = 0$        $y(e) = e$

### ANSWER (Section–A)

1. Order : 1, degree : 2
2. Order : 3, degree : not defined
3. Order : 4, degree : not defined
4. Order : 3, degree : 1
5. Order : 2, degree : 1

### ANSWER (Section–B :)

1.  $(x^2 - 9)(y^1)^2 + x^2 = 0$
2.  $2xy y' + x^2 = y^2$
3.  $xyy'' + x(y^1)^2 - yy' = 0$
4.  $y = x \sin^{-1} x + \sqrt{1-x^2} + c$
5.  $\tan y = c(1 - e^x)$
6.  $\tan x \tan y = c$
7.  $y = \frac{1}{4} \log \left[ (x+1)^2 (x^2+1)^3 \right] - \frac{1}{2} \tan^{-1} x + 1$
8.  $y - x - 2 = \log \left[ x^2 (y+2)^2 \right]$

$$9- y = \left( \frac{\text{Sin}x - \text{Cos}x}{2} \right) + ce^x$$

$$10- y = x^2 - \frac{\pi^2}{4\text{Sin}x} (\text{Sin}x \neq 0)$$

$$11- y = \frac{x^2}{4} + cx^{-2}$$

$$12- y \log x = \frac{-2}{x} (1 + \log x) + c$$

$$13- y = \frac{1}{x} - \text{Cot}x + \frac{c}{x\text{Sin}x}$$

$$14- y = 4\text{Sin}^3x - 2\text{Sin}^2x$$

$$15- y\text{Sec}x = x^2\text{Sec}x + c$$

$$16- x = \tan(x + y) - \text{Sec}(x + y) + c$$

$$17- \frac{1}{2} \tan^{-1} \left( \frac{4x + y + 1}{2} \right) = x + c$$

$$18- x - y = e^{x+y+1}$$

### Answer (Section-C)

$$1. y = \frac{2x}{3x^2 - 1}$$

$$2. \frac{-x^3}{3y^3} + \log|y| = c$$

$$3. \left( y + \sqrt{x^2 + y^2} \right)^2 = c^2 x^4$$

$$4. \log|x| = \text{Cos} \left( \frac{y}{x} \right)$$

$$5. x\text{Sin} \left( \frac{y}{x} \right) = c$$

$$7. y = \frac{2x}{1 + \log|x|}$$

## VECTOR ALGEBRA

### Important definition and facts :-

1. The unit vector in direction of vector  $\vec{a}$  is given by  $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$
2. Two vectors  $\vec{a}$  and  $\vec{b}$  are colinear if and only if  $\vec{a} = l\vec{b}$ ,  $l$  being a scalar
3. Scaler or Dot product of two vectors :-  
The scaler or dot product of two vectors  $\vec{a}$  and  $\vec{b}$  denoted by  $\vec{a} \cdot \vec{b}$  is given by.  
 $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$ .  
where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ ,  $0 \leq \theta \leq \pi$
4. Property of of scaler product :-  
Property-1 :- The scaler product of two vectors is commutatives  
*i.e.*  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$   
Property-2 :- The scaler product of vector is distributive over addition *i.e.*  
(i)  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$  (ii)  $(\vec{b} + \vec{c}) \cdot \vec{a} = \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}$   
Property-3 :- Let  $\vec{a}$  and  $\vec{b}$  be two non zero vectors then  $\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$   
Property-4 :- Let  $\vec{a}$  and  $\vec{b}$  be any two vectors and  $\lambda$  be any scalar then  
 $(\lambda \vec{a}) \cdot \vec{b} = \lambda (\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\lambda \vec{b})$
5. Angle between two vectors :- Let  $\theta$  be the angle between two vectors  $\vec{a}$  and  $\vec{b}$  then.  
 $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$   
$$\Rightarrow \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \Rightarrow \theta = \cos^{-1} \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$$
6. Projection of  $\vec{a}$  on the vector  $\vec{b}$  is given by.  
 $\vec{a} \cdot \hat{b}$  or  $\vec{a} \cdot \left( \frac{\vec{b}}{|\vec{b}|} \right)$  or  $\frac{1}{|\vec{b}|} (\vec{a} \cdot \vec{b})$
7. If  $\theta = 0$  then projection vector of  $\vec{a}$  will be  $\vec{a}$  and if  $\theta = \pi$  then projection vector of  $\vec{a}$  will be  $-\vec{a}$
8. Vector ( or cross product) of two vectors :-  
The vector or cross product of two non parallel (non-collinear) vector  $\vec{a}$  and  $\vec{b}$  denoted by  $\vec{a} \times \vec{b}$  is defined as  $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$ ,  $0 < \theta < \pi$  and  $\hat{n}$  is two unit vector perpendicular to both  $\vec{a}$  and  $\vec{b}$
9. Property of vector product :-  
(i)  $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ , However  $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

$$(ii) \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

(iii) If  $\vec{a}$  and  $\vec{b}$  are parallel (or collinear) then  $\vec{a} \times \vec{b} = 0$

(iv)  $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$  where m is scalar.

10. For mutually perpendicular unit vector  $\hat{i}, \hat{j}, \hat{k}$ . we have,

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$$

11. Angle between two vectors  $\vec{a}$  and  $\vec{b}$  may be given as  $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

12. Area of triangle whose sides are along  $\vec{a}$  and  $\vec{b}$  is given by  $\frac{1}{2}|\vec{a} \times \vec{b}|$

13. If  $\vec{a}$  and  $\vec{b}$  represent the adjacent sides of parallelogram. Then its area is given by  $|\vec{a} \times \vec{b}|$

14. Unit vector  $\hat{n} = \pm \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

15. Let  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ ,  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$  then  $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

### Illustrative Examples :

1. If  $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ ,  $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$  and  $\vec{c} = 3\hat{i} + \hat{j}$  find  $\lambda$  such that  $\vec{a} + \lambda\vec{b}$  is perpendicular to  $\vec{c}$ .

Solution :

$$\text{We have } \vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}, \vec{c} = 3\hat{i} + \hat{j}$$

$$\vec{a} + \lambda\vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$$

$$= (1-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$$

$$\because (\vec{a} + \lambda\vec{b}) \perp \vec{c} \Rightarrow (\vec{a} + \lambda\vec{b}) \cdot \vec{c} = 0$$

$$\therefore (1-\lambda)3 + (2+2\lambda) \cdot 1 + (3+\lambda) = 0 \Rightarrow \lambda = 5$$

2. Find the value of constant  $\lambda$  such that scalar product of the vector  $\hat{i} + \hat{j} + \hat{k}$  with the unit vector parallel to the sum of vector  $2\hat{i} + 4\hat{j} - 5\hat{k}$  and  $\hat{i} + 2\hat{j} + 3\hat{k}$  is equal to one. Solve :

$$\vec{b} + \vec{c} = (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$

$$\therefore |\vec{b} + \vec{c}| = \sqrt{(2 + \lambda)^2 + (6)^2 + (2)^2} = \sqrt{\lambda^2 + 4\lambda + 44}$$

$$\therefore \text{unit vector parallel to sum of the vector } \vec{b} \text{ and } \vec{c} = \frac{\vec{b} + \vec{c}}{|\vec{b} + \vec{c}|}$$

$$\therefore \text{As per question } \vec{a} \cdot \frac{(\vec{b} + \vec{c})}{|\vec{b} + \vec{c}|} = 1$$

$$(\hat{i} + \hat{j} + \hat{k}) \cdot \frac{[(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}]}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$$\therefore 2 + \lambda + 6 - 2 = \sqrt{\lambda^2 + 4\lambda + 44}$$

$$(\lambda + 6)^2 = \lambda^2 + 4\lambda + 44$$

$$\therefore 8\lambda = 8 \Rightarrow \lambda = 1$$

3. If  $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$  and  $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$  show  $(\vec{a} - \vec{d})$  is parallel to  $(\vec{b} - \vec{c})$  solve :

$$\begin{aligned} (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) &= \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{d} \times \vec{b} + \vec{d} \times \vec{c} \\ &= \vec{a} \times \vec{b} - \vec{b} \times \vec{d} + \vec{b} \times \vec{d} - \vec{a} \times \vec{b} \\ &= 0 \end{aligned}$$

4. If  $\vec{a}, \vec{b}, \vec{c}$  are the position vector of the non collinear point A,B,C respectively in space show that  $\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}$  is perpendicular to plane ABC

**Solution :** Let  $O\vec{A} = \vec{a}, O\vec{B} = \vec{b}, O\vec{C} = \vec{c}$

$$\therefore \vec{AB} = \vec{b} - \vec{a}, \vec{AC} = \vec{c} - \vec{a}$$

$$\begin{aligned} \text{New } \vec{AB} \times \vec{AC} &= (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) \\ &= \vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a} \\ &= \vec{b} \times \vec{c} + \vec{b} \times \vec{a} \times \vec{b} + \vec{c} \times \vec{a} + 0 \end{aligned}$$

$$\therefore \vec{AB} \times \vec{AC} \text{ is perpendicular to } \vec{AB} \text{ and } \vec{AC}$$

$$\therefore \vec{AB} \times \vec{AC} \text{ i.e. } \vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b} \text{ is perpendicular to plane ABC}$$

5. Find the area of parallelogram whose diagonals are  $4\hat{i} - \hat{j} - 3\hat{k}$  and  $-2\hat{i} + \hat{j} - 2\hat{k}$

**Solution :**

$$\begin{aligned} \text{Area} &= |\vec{a} \times \vec{b}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & -3 \\ -2 & 1 & -2 \end{vmatrix} \\ &= \hat{i}(2+3)\hat{j}(-8-6) + \hat{k}(4-2) \\ &= 5\hat{i} + 14\hat{j} + 2\hat{k} \end{aligned}$$

6. Show that  $(\vec{a} \times \vec{b})^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$

**Solution :**

$$\begin{aligned} (\vec{a} \times \vec{b})^2 &= |\vec{a} \times \vec{b}|^2 \\ (\vec{a} \times \vec{b})^2 &= \{|\vec{a}||\vec{b}|\sin\theta\}^2 \\ (\vec{a} \times \vec{b})^2 &= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta) \\ (\vec{a} \times \vec{b})^2 &= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta \\ &= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})(\vec{a} \cdot \vec{b}) \\ &= \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix} \end{aligned}$$

### PRACTICE EXERCISE (Section-A)

1. If  $\vec{a}$  is any vector then find the value of  $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$
2. If  $|\vec{a} \times \vec{b}| = 4, |\vec{a} \cdot \vec{b}| = 2$ , find  $|\vec{a}|^2 |\vec{b}|^2$
3. Find the value of  $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \cdot \hat{j})$
4. If  $\vec{b} \times \vec{c} = \vec{c} \times \vec{a} \neq 0$  then prove that  $\vec{a} + \vec{b} = k\vec{c}$
5. If  $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$  show that  $\vec{a}$  and  $\vec{b}$  are perpendicular.
6. Find the unit vector in the direction of  $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$
7. If  $x(\hat{i} + \hat{j} + \hat{k})$  is a unit vector find the value of  $x$

8. Find the unit vector perpendicular to both  $\hat{i} + \hat{j}$  and  $\hat{j} + \hat{k}$
9. Prove  $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$
10. Find the area of triangle whose vertices are  $A(3, -1, 2)$ ,  $B(1, -1, -3)$  and  $C(4, -3, 1)$
11. If  $|\vec{a}| = \sqrt{26}$ ,  $|\vec{b}| = 7$  and  $|\vec{a} \times \vec{b}| = 35$  find  $|\vec{a} \cdot \vec{b}|$ .
12. If  $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ ,  $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$  show that  $\vec{a} + \vec{b}$  is perpendicular to  $\vec{a} - \vec{b}$ .

### SECTION - B

1. If  $\hat{a}$  and  $\hat{b}$  are unit vector inclined at angle  $\alpha$ . Then prove that  $\sin \frac{\theta}{2} = \frac{1}{2} |\vec{a} - \vec{b}|$
2. Find a unit vector perpendicular to each of the vector  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$  where  $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$  and  $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ .
3. If  $\vec{a} + \vec{b} + \vec{c} = 0$   
Show that  $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$
4. If  $\vec{a}$  is any vector in space. Show that  $\vec{a} = (\vec{a} \cdot \hat{i})\hat{i} + (\vec{a} \cdot \hat{j})\hat{j} + (\vec{a} \cdot \hat{k})\hat{k}$ .
5. If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are unit vector such that  $\vec{a} + \vec{b} + \vec{c} = 0$  then find the value of  $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ .
6. Two adjacent side of parallelogram are  $2\hat{i} - 4\hat{j} + 5\hat{k}$  and  $\hat{i} - 2\hat{j} - 3\hat{k}$ . Find the unit vector perpendicular to its diagonal. Also, find its area.
7. If  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are mutually perpendicular unit vector then find  $|\vec{a} + \vec{b} + \vec{c}|$ .
8. Show that the vectors  $2\hat{i} - \hat{j} + \hat{k}$ ,  $\hat{i} - 3\hat{j} + 5\hat{k}$  and  $3\hat{i} - 4\hat{j} - 4\hat{k}$  forms the vertices of a right angled triangle. Also find the area.
9. If  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ ,  $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ ,  $\vec{a} \neq 0$  then show that  $\vec{b} = \vec{c}$ .
10. If  $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ ,  $\vec{a} \neq 0$  and  $\vec{b} \neq \vec{c}$  show that  $\vec{b} = \vec{c} + t\vec{a}$  for some scalar  $t$ .
11. Prove that the points  $A, B$  and  $C$  with position vector  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  respectively are collinear if and only if  $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$ .
12. Find a unit vector perpendicular to the plane ABC where  $A, B, C$  are the points  $(3, -1, 2)$ ,  $(1, -1, -3)$  and  $(4, -3, 1)$  respectively.

### SECTION - C

1. If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are the position vector of the vertices  $A, B, C$  of a triangle  $ABC$ . Show that the area of triangle  $ABC$  is  $\frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$ . Deduce the condition for points  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  to be collinear.

2. For any vector  $\vec{a}$  prove that  $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2|\vec{a}|^2$ .
3. If  $a, b, c$ , are the length of sides  $BC, CA$  and  $AB$  of a triangle  $ABC$ . Prove that  $\vec{BC} + \vec{CA} + \vec{AB} = 0$  and deduce that  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ .
4. If  $\vec{\alpha} = 3\hat{i} - \hat{j}$ ,  $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$ , then express  $\vec{\beta}$  in the term of  $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$  where  $\vec{\beta}_1$  is parallel to  $\vec{\alpha}$  and  $\vec{\beta}_2$  is perpendicular to  $\vec{\alpha}$ .
5. Let  $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$ ,  $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$  and  $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$  find a vector  $\vec{d}$  which is perpendicular to both  $\vec{a}$  and  $\vec{b}$  and  $\vec{c} \cdot \vec{d} = 15$
6. Show that distance of the point  $\vec{c}$  from the line joining  $\vec{a}$  and  $\vec{b}$  is  $\frac{|\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}|}{|\vec{b} - \vec{a}|}$
7. Show that the angle between two diagonal of a cube is  $\cos^{-1}\left(\frac{1}{3}\right)$ .
8. If  $\vec{a}, \vec{b}, \vec{c}$  are three mutually perpendicular vectors of equal magnitude, prove that  $\vec{a} + \vec{b} + \vec{c}$  is equally inclined with vectors  $\vec{a}, \vec{b}$ , and  $\vec{c}$ .

### ANSWER (SECTION-A)

1.  $2\vec{a}^2$ ;    2. 20;    3. 1;    6.  $\frac{2l+3j+k}{\sqrt{14}}$     7.  $+\frac{1}{\sqrt{3}}$
8.  $\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$     10.  $\frac{1}{2}\sqrt{165}$     11.  $\pm 7$

### Section - B

2.  $\pm \frac{2}{3}(\hat{i} - \hat{j} - \hat{k})$     4.  $\frac{-3}{2}$     6.  $\frac{1}{7}(3\hat{l} - 6\hat{j} + 2\hat{k}), 11\sqrt{5}$     7.  $\sqrt{3}$
12.  $\frac{1}{\sqrt{165}}(-10\hat{l} - 7\hat{j} + 4\hat{k})$

### Section - C

4.  $\vec{B}_1 = \frac{3}{2}\hat{l} - \frac{1}{2}\hat{j}$  and  $\vec{B}_2 = \frac{1}{2}\hat{l} - \frac{3}{2}\hat{j} - 3\hat{k}$
5.  $\frac{1}{3}(160\hat{i} - 5\hat{j} + 70\hat{k})$

## THREE DIMENSIONAL GEOMETRY

**\* Important Definition and Result :-**

1. If  $l, m, n$  are the direction cosine of a vector, then  $l^2 + m^2 + n^2 = 1$ .
2. If  $a, b, c$  are the direction ratio of a vector, then its direction cosine are given by

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

3. If  $\theta$  is the angle between two vector whose direction cosines are  $(l_1, m_1, n_1)$  and  $(l_2, m_2, n_2)$  then  $\text{Cos}\theta = l_1l_2 + m_1m_2 + n_1n_2$

4. Two vector having direction cosine  $(l_1, m_1, n_1)$  and  $(l_2, m_2, n_2)$  are orthogonal iff  $l_1l_2 + m_1m_2 + n_1n_2 = 0$ .

5. Parallel vector have equal direction ratio

6. The vector equation of a st. line through fixed point with position vector  $\vec{a}$  and parallel to  $\vec{b}$  is  $\vec{r} = \vec{a} + \lambda\vec{b}$  where  $\lambda$  is a scalar.

7. The cartesian equation of a line with direction ratio  $a, b, c$ , and passing through  $(x_1, y_1, z_1)$  is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

8. The coordinates of any point on the line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda \text{ are } (x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$$

9. Vector equation of line passing through two points with position vector  $\vec{a}$  and  $\vec{b}$   $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

10. The cartesian equation of line through two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

11. Angle between lines  $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$  is given by  $\text{Cos}\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| |\vec{b}_2|}$

12. If lines are  $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$  and  $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$  Then,

$$\text{Cos}\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

13. The condition for orthogonality is  $\vec{b}_1 \cdot \vec{b}_2 = 0$  or  $a_1a_2 + b_1b_2 + c_1c_2 = 0$

14. If  $d$  is the shortest distance between two lines

$$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1 \text{ and } \vec{r} = \vec{a}_2 + \lambda\vec{b}_2 \text{ then } d = \frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_1 - \vec{a}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$$

15. If two lines are intersection then  $(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 \times \vec{a}_1) = 0$  and in cartesian form is

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

16. Distance between two parallel lines  $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{b}_1$  is given by  $\frac{|\vec{b}_1 \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1|}$

17. General equation of plane in vector form is  $\vec{r} \cdot \vec{\eta} = d$

18. General equation of plane in cartesian form is  $ax + by + cz + d = 0$  where a,b,c are the D.R's of normal to the plane.

19. Equation of plane passing through a point with position vector  $\vec{a}$  is  $(\vec{r} - \vec{a}) \cdot \vec{\eta} = 0$

20. Equation of plane passing through a point  $(x_1, y_1, z_1)$  is  $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

21. If  $\theta$  is the angle between two planes  $\vec{r} \cdot \vec{\eta}_1 = d_1$  and  $\vec{r} \cdot \vec{\eta}_2 = d_2$  then  $\cos \theta = \frac{|\vec{\eta}_1 \cdot \vec{\eta}_2|}{|\vec{\eta}_1| |\vec{\eta}_2|}$

22. In Cartesian form  $\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$

23. Two planes are perpendicular then  $\vec{n}_1 \cdot \vec{n}_2 = 0$

24. Two Planes are parallel then  $\vec{n}_1 \times \vec{n}_2 = 0$

25. If  $\theta$  is the angle between line  $\vec{r} = \vec{a} + \vec{m}$  and plane  $\vec{r} \cdot \vec{n} = d$  then  $\sin \theta = \frac{|\vec{m} \cdot \vec{n}|}{|\vec{m}| |\vec{n}|}$

26. Distance of a plane  $\vec{r} \cdot \vec{n} = d$  from a point with position vector  $\vec{a}$  is  $\frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}$

27. Distance of a plane  $ax + by + cz + d = 0$  from a point  $(x_1, y_1, z_1)$  is  $\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$

28. Two lines  $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$  and  $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$  are coplanar iff

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

and equation of plane containing them is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

**Illustrative Example :**

1. Find the direction cosine of the line  $\frac{x-2}{2} = \frac{2y-5}{-3}, z = -1$ . Also find the vector equation of line.

**Solution :**

Given line can be written as

$$\Rightarrow \frac{x-2}{2} = \frac{2y-5}{-3} = \frac{z+1}{0}$$

$$\Rightarrow \frac{x-2}{2} = \frac{y-\frac{5}{2}}{-\frac{3}{2}} = \frac{z+1}{0}$$

$\therefore$  D.R. are  $2, -\frac{3}{2}, 0$

Direction cosines are  $\frac{2}{\sqrt{2^2 + (-\frac{3}{2})^2 + 0^2}}, \frac{-\frac{3}{2}}{\sqrt{2^2 + (-\frac{3}{2})^2 + 0^2}}, \frac{0}{\sqrt{2^2 + (-\frac{3}{2})^2 + 0^2}}$

or,  $\frac{4}{5}, -\frac{3}{5}, 0$

given line passing through the point having position vector  $\vec{a} = 2\hat{i} + \frac{5}{2}\hat{j} - \hat{k}$  and is parallel to  $\vec{b} = 2\hat{i} - \frac{3}{2}\hat{j} - 0\hat{k}$

$\therefore$  vector eq<sup>n</sup> is  $\vec{r} = (2\hat{i} + \frac{5}{2}\hat{j} - \hat{k}) + \lambda(2\hat{i} - \frac{3}{2}\hat{j} - 0\hat{k})$

2. Find the coordinate up foot of perpendicular drawn from the point A (1,2,1) to the line joining B (1,4,6) and C (5,4,4)

**Solution :**

Direction ratio of BC are  $5-1, 4-4, 4-6$  or,  $2, 0, -1$

$\therefore$  equation of line BC are  $\frac{x-1}{2} = \frac{y-4}{0} = \frac{z-6}{-1}$

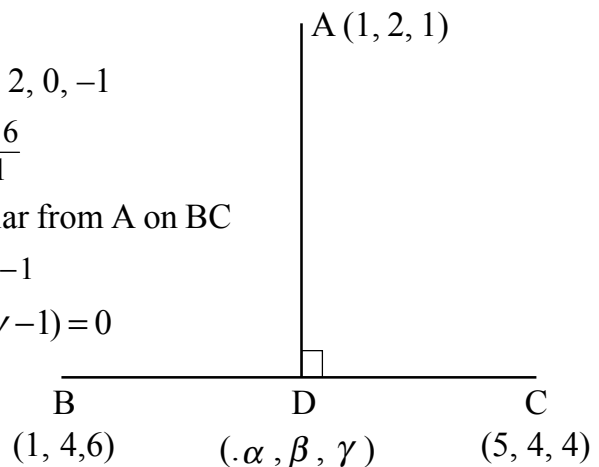
Let D ( $\alpha, \beta, \gamma$ ) be the foot of perpendicular from A on BC

$\therefore$  Direction ratio of AD are  $\alpha-1, \beta-2, \gamma-1$

$\therefore AD \perp BC \Rightarrow 2(\alpha-1) + 0(\beta-2) + (-1)(\gamma-1) = 0$

$\therefore 2\alpha - \gamma = 0 \rightarrow (1)$

$\therefore D(\alpha, \beta, \gamma)$  lie on the line BC then



$$\therefore \frac{\alpha - 1}{2} = \frac{\beta - 4}{0} = \frac{\gamma - 6}{-1} = \lambda \quad (\text{say})$$

$$\therefore \alpha = 2\lambda + 1, \beta = 4, \gamma = -\lambda + 6$$

$$\text{from (1)} \Rightarrow 2(2\lambda + 1) - (-\lambda + 6) = 1$$

$$\therefore \lambda = 1$$

$$\therefore \alpha = 3, \beta = 4, \gamma = 5$$

Hence, foot of perpendicular  $= (\alpha, \beta, \gamma) = (3, 4, 5)$

3. Find the angle between line  $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$  and the plane

$$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) = 4$$

**Solution :**

$$\text{Here, } \vec{b} = \hat{i} - \hat{j} + \hat{k} \text{ and } \vec{\eta} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\therefore \sin\theta = \frac{(\hat{i} - \hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{2^2 + 1^2 + 1^2}} = \frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}$$

$$\therefore \theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

4. Find the equation of plane containing the lines at intersection of the planes  $x + y + z = 6$  and  $2x + 3y + 4z + 5 = 0$  and passing through the point  $(1, 1, 1)$

**Solution :**

Equation of plane through the intersection of two gicent planes is

$$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0 \text{ it passes through } (1, 1, 1)$$

$$\therefore -3 + 14\lambda = 0 \Rightarrow \lambda = \frac{3}{14}$$

$\therefore$  required equation of plane is –

$$(x + y + z - 6) + \frac{3}{14}(2x + 3y + 4z + 5) = 0$$

$$\text{or, } 20x + 23y + 26z - 69 = 0$$

5. Show that the lines –

$$\vec{r} = (2\hat{j} - 3\hat{k}) + \lambda(\hat{l} + 2\hat{j} + 3\hat{k}) \text{ and}$$

$$\vec{r} = (2\hat{i} - 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k}) \text{ are coplanar. Find the equation plane}$$

containing them.

**Solution :** We know line  $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$  are coplanar if  $[\vec{a}_2 - \vec{a}_1, \vec{b}_1, \vec{b}_2] = 0$

$$\vec{a}_1 = 2\hat{j} - 3\hat{k}, \quad \vec{a}_2 = 2\hat{j} + 6\hat{j} + 3\hat{k}$$

$$\vec{b}_1 = \hat{i} - 2\hat{j} + 3\hat{k}, \quad \vec{b}_2 = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\therefore \begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 3 \end{vmatrix} = 0 \quad \therefore \text{given lines are coplanar equation of plane containing them is given}$$

by-

$$[\vec{r}_2 - \vec{a}_1, \vec{b}_1, \vec{b}_2] = 0 \Rightarrow x - 2y + z + 7 = 0 \text{ and in vector form } \vec{r} \cdot (\hat{i} + 2\hat{j} + \hat{k}) + 7 = 0$$

6. Find the image of the point having position vector  $\hat{i} + 3\hat{j} + 4\hat{k}$  in the plane

$$\vec{r} \cdot (2\hat{i} + \hat{j} + \hat{k}) + 3 = 0$$

**Solution :**

Let Q be the image of the point  $p(\hat{i} + 3\hat{j} + 4\hat{k})$  in the plane  $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$

Therefore equation of line PQ is  $\vec{r} = (\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$

$\therefore$  Q lie on the line PQ, so let the P.V. of Q be  $(1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (4 + \lambda)\hat{k}$

$\therefore$  R is the mid point of PQ therefore P.V. of R is

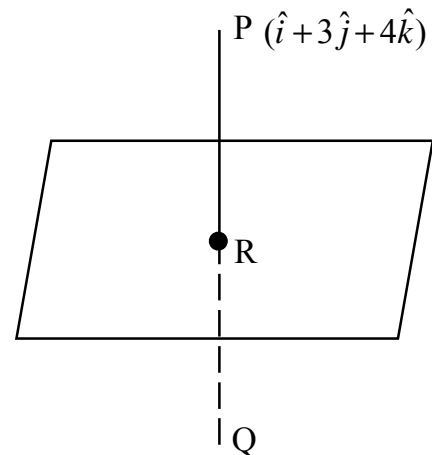
$$\frac{[(1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (4 + \lambda)\hat{k}] + [\hat{i} + 3\hat{j} + 4\hat{k}]}{2}$$

$$= (\lambda + 1)\hat{i} + (3 - \lambda/2)\hat{j} + (4 + \lambda/2)\hat{k}$$

Q Lie on the plane  $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$

$$\{(\lambda + 2)\hat{i} + (3 - \lambda/2)\hat{j} + (4 + \lambda/2)\hat{k}\} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$$

$$\Rightarrow 2\lambda + 2 - 3 + \lambda/2 + 4 + \lambda/2 + 3 = 0$$



$$\therefore \lambda = -2$$

Thus P.V. of Q is  $(\hat{i} + 3\hat{j} + 4\hat{k}) - 2(2\hat{i} - \hat{j} + \hat{k})$

$$\Rightarrow -3\hat{i} + 5\hat{j} + 2\hat{k}$$

7. Find the length of foot of perpendicular from the point (7, 14, 5) to the plane  $2x+4y-z=z$

**Solution :**

D. r. of normal to plane  $2x+4y-z=z$  are 2, 4, -1

Let Q ( $\alpha, \beta, \lambda$ ) be the foot of perpendicular from (7, 14, 5)

$\therefore$  D.  $\gamma$ . S. of PQ are  $\alpha-7, \beta-14, \gamma-5$

$\therefore$  P $\theta$  and normal to plane are parallel

$$\frac{\alpha-7}{2} = \frac{\beta-14}{4} = \frac{\gamma-5}{-1} = K \text{ (say)}$$

$\therefore \alpha = 2K + 7, \beta = 4K + 14, \gamma = -k + 5$

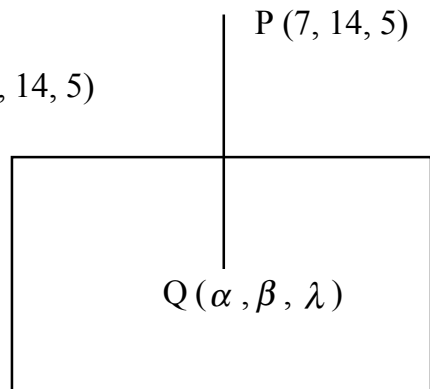
$\therefore$  Q Lie on the plane  $2x+4y-z=z$

$\therefore 2(2K+7)+14(4K+4) - (-K+5) = 2$

$\therefore K = -3$

$\therefore$  foot of perpendicular is (1, 2, 8)

$$\therefore \text{Length of PQ} = \sqrt{(7-1)^2 + (14-2)^2 + (5-\theta)^2} = 3\sqrt{21}$$



8. A Variable plane which remain at constant distance 3p from origin eat the coordinate axis at A, B, C. Show that the locus ao the centroid of triangle ABC is  $x^2+y^2+z^2 = P^2$

**Solution :**

Let equation of place be  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ .....(1) where a,b,c, are variable

This meets x,y,z axes at A (a,0,0), B (0,b,0), C (0,0,c)

Let ( $\alpha, \beta, \gamma$ ) be the centroid of triangle ABC

$$\therefore \alpha = \frac{a+0+0}{3} = \frac{a}{3}, \beta = \frac{0+b+0}{3} = \frac{b}{3}$$

$$r = \frac{0+0+c}{3} = \frac{c}{3}$$

The plane i at distance 3p from origin

$$\therefore 3p = \frac{\left| \frac{o}{a} + \frac{o}{b} + \frac{o}{c} - 1 \right|}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}}$$

$$\therefore 3p = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$\frac{1}{9p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

from (ii) we have,  $a = 3\alpha, b = 3\beta, c = 3\gamma$

$$\frac{1}{9p^2} = \frac{1}{9\alpha^2} + \frac{1}{9\beta^2} + \frac{1}{9\gamma^2}$$

or, 
$$\frac{1}{p^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$$

$\therefore$  locus of  $(\alpha, \beta, \gamma)$  is

$$\begin{aligned} \frac{1}{p^2} &= \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \\ \Rightarrow p^{-2} &= x^{-2} + y^{-2} + z^{-2} \end{aligned}$$

### PRACTICE EXERCISE (SECTION-A)

1. Find the angle between  $2x = 3y = -z$  and  $6x = -y = -4z$
2. If a line makes  $30^\circ, 60^\circ, 90^\circ$  with position direction of x,y and z axis find its direction cosine
3. Find the direction cosine of a line which makes equal angle with coordinate axis.
4. Find the intercept cut off by the plane  $3x + 6y - 3z = 7$
5. Find the value of  $\lambda$  so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2\lambda} = \frac{z-3}{2} \text{ and } \frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5} \text{ are at right angle}$$

6. Find the equation of line through  $(-1, 2, 1)$  and parallel to  $\frac{2x-1}{4} = \frac{3y+5}{2} = \frac{2-z}{3}$
7. What is the direction cosine of x, y and z axis
8. Find the vector equation of plane passes through the point  $(1, 0, -2)$  and normal to the plane  $(\hat{i} + \hat{j} - \hat{k})$
9. Find the distance between two plans  $2x + 3y + 4z = 4$  and  $4x + 6y + 8z = 12$

10. Find the coordinant of the point where the line through (5,1,6) and (3,4,1) crosses the yz– plane
11. Write the equation of plane parallel to xoy plane and passing through the point (2,-3,5)
12. Write the distance of plane  $\vec{\gamma} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 12$  from origin
13. Write equation of plane passing thropugh (a,o,o), (o,b,o) and (o,o,c)

### SECTION–B

1. Find the image of the point  $(\hat{i} + 3\hat{j} + 4\hat{k})$  in the plane  $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$
2. A line makes angle  $\alpha, \beta, \gamma$  and  $\delta$  with the diagonal of cube prove that  $\text{Cos}^2\alpha + \text{Cos}^2\beta + \text{Cos}^2\gamma + \text{Cos}^2\delta = \frac{4}{3}$
3. Find the image of the point (3,-2,1) in the plane  $3x - y + 4z = 2$
4. Find the image of the (1,6,3) in the line  $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$
5. Find the lenght of foot of perpendicular from the point (1,1,2) to the plane  $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$
6. Find the equation of the plane through the points (2,1,0), (3,-2,-2) and (3,1,7)
7. Find the equation of the plane which contains the line of interesection of the planes  $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 4$  and  $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$  which is perpendicular to the plane  $\vec{\gamma} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + \beta = 0$
8. Find the equation of the plane passing through the point (-1,3,2) and perpendicular to each of the plane  $x + 2y + 3z = 5$  and  $3x + 3y + z = 0$
9. Find the distance of point with p.v.  $-\hat{i} - 5\hat{j} - 10\hat{k}$  from the point of interesection of the line  $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 12\hat{k})$  with the plane  $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$
10. Find the angle between lines  $x - 2y + z = 0 = x + 2y - 2z$  and  $x + 2y + z = 0 = 3x + 9y + 5z$
11. Find the equation of plane through the points (1,0,-1) (3,2,2) and parallel to the line  $\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-2}{3}$
12. If a variable plane of a constant distance p from the origin meets the coordinant areas in point A,B and C respectively through these points, planes are drawn parallel to the coordinant plane show that locas of the point of interesection is  $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$
13. Find the equation of plane through the interesection of planes  $\vec{r} \cdot (\hat{i} + 3\hat{j}) + 6 = 0$

and  $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$  which is at a unit distance from the origin.

### SECTION - C

- Find the shortest distance between the lines given by  $\vec{r} = (3\hat{i} + 8\hat{j} + 8\hat{k}) + r(-3\hat{i} + 2\hat{j} + 4\hat{k})$  and  $\vec{r} = (-3\hat{i} - 7\hat{j} + 6\hat{k}) + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$  also find the equation of shortest distance.
- Show that two lines  $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$  and  $\frac{x-4}{5} = \frac{y-1}{2} = z$  intersect. Also find the point of intersection of these lines.
- A variable plane is at a constant distance  $p$  from the origin and meets coordinate axes in A, B, C. Show that the locus of the centroid of the tetrahedron  $\Delta ABC$  is  $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$
- Shows that the lines  $\frac{x-1}{2} = \frac{1-y}{2} = -z$  and  $\frac{x-4}{3} = \frac{1-y}{2} = z-1$  are coplanar. Also find the equation of plane containing them.
- Show that the lines  $\vec{r} = (2\hat{i} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$  and  $\vec{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$  are coplanar. Also, find the equation of the plane containing them.
- Find the vector and cartesian equation line passing through the point  $(\hat{i} + 2\hat{j} + 3\hat{k})$  and parallel to planes  $\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 5$  and  $\vec{r} \cdot (3\hat{i} + \hat{j} + \hat{k}) = 6$
- Find the distance of point  $(-1, -5, -10)$  from the point of intersection of the line  $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$  and plane  $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$ .

### ANSWER (SECTION - A)

- $90^\circ$ ,  $2. \frac{\sqrt{3}}{2}, \frac{1}{2}, 0$   $3. \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}$
- $\frac{7}{3}, \frac{7}{6}, \frac{-7}{3}$   $5. \lambda = \frac{70}{11}$   $6. \frac{x+1}{2} = \frac{y-2}{2/3} = \frac{z-1}{-3}$
- $(1, 0, 0)$   $8. [\vec{r} - (\hat{i} - 2\hat{k})] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0$   $9. \frac{2}{\sqrt{29}}$   
 $(0, 0, 1)$   
 $(1, 0, 0)$

10.  $\left(0, \frac{17}{2}, -\frac{13}{2}\right)$       11.  $z = 5$     12. 4    13.  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

**SECTION - B**

1. (3, 1, 6)      3. (0, -1, -3)      4. (1, 0, 7)      5.  $\frac{13\sqrt{6}}{12}$   
 6.  $7x + 3y - z = 17$       7.  $33x + 45y + 50z - 41 = 0$       8.  $7x - 8y + 3z + 25 = 0$

10.  $\theta = \cos^{-1}\left(\frac{8}{\sqrt{406}}\right)$     11.  $4x - y - 2z - 6 = 0$

13.  $\vec{r} \cdot (-2\hat{i} + 4\hat{j} + 2\hat{k}) + 6 = 0$     or     $\vec{r} \cdot (4\hat{i} + 2\hat{j} - 4\hat{k}) + 6 = 0$

**SECTION - C**

1.  $S.D = 3\sqrt{30}$ ,  $\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \mu(-6\hat{i} - 15\hat{j} + 3\hat{k})$     2. (-1, -1, -1)

4.  $2x - 5y - 16z + 13 = 0$     5.  $\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) + 7 = 0$

6.  $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$     7. 13 unit.

# LINEAR PROGRAMMING

## Important definition and facts :

1. Linear programming is a tool which is used in decision making in business for obtaining maximum and minimum value of quantity subject to certain constraints.
2. Objective function :- The linear function  $z = c_1x_1 + \dots + c_nx_n$  which is to be maximized (or minimized) is called the objective function of L.P.P.
3. Feasible solution :- Any solution to the general L.P.P. which satisfy the non-negative restriction of the problem is called a feasible solution to the general L.P.P.
4. Optimum solution :- Any feasible solution which optimise (minimizes or maximizes) the objective function of a general L.P.P. is called an optimum solution

## Important facts :

- (i) Feasible region is the intersection of the regions of the inequation.
- (ii) The feasible region is always a convex polygon
- (iii) Iso profit lines are parallel to the objective function
- (iv) The objective function is maximum or minimum at a point, which lines on the boundary of the feasible region.

## ILLUSTRATIVE EXAMPLE

1. Minimized  $f = 6x + 7y$  subject to  $10x + 4y \geq 20, 5x + 5y \geq 20, 2x + 6y \geq 12, x, y \geq 0$

### Solution :

(i)  $10x + 4y = 20$

$5x + 2y = 10$  clearly, point A (0, 5) and B (2, 0) lie on the this line (0, 4) and (non-origin side)

(ii)  $5x + 5y = 20$

$x + y = 4$  clearly, p(0, 4) and Q (4, 0) lie on this line and (non-origin)

(iii)  $2x + 6y = 12$

$x + 3y = 6$   
clearly, the points L (0, 2) and m (6, 0) lie on this line and (non- origin)

The points of the feasible region are m, T, S and A where S is point of Inter section of line AB and PQ

$x + y = 4$  and  $5x + 2y = 10$  gives  $x = 2/3, y = 10/3$

$\therefore S = (2/3, 10/3)$

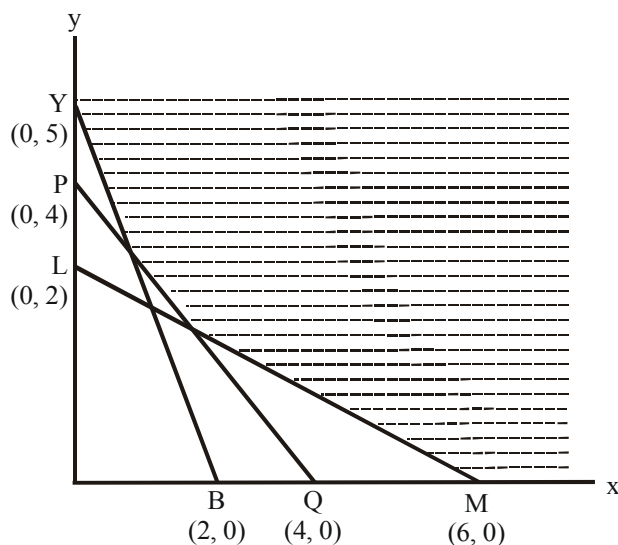
T is the point of intersection of LM and PQ  $x + y = 4$  and  $x + 3y = 6$  gives  $x = 3, y = 1$

$\therefore T = (3, 1)$

At M (6, 0),  $f = 6 \times 6 + 7 \times 0 = 36$

At T (3, 1),  $f = 6 \times 3 + 7 \times 1 = 25$

At S (2/3, 10/3),  $f = 6 \times 2/3 + 7 \times 10/3 = 82/3$



At A (0, 5),  $f = 35$

∴ Optimum solution of given problems corresponds to T (3, 1)

∴ Minimum value is 25 when  $x = 3, y = 1$

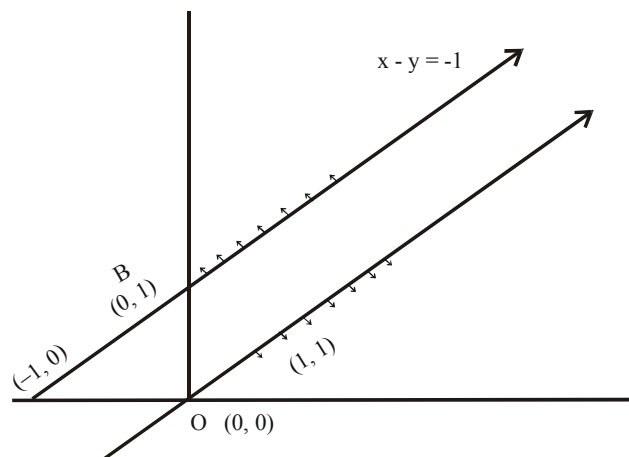
2. Maximize  $z = x + y$  subject to  $x - y \leq -1$   $-x + y \leq 0$ ,  $x, y \geq 0$

**Solution :**

(i)  $x - y = -1$  which passes through the points A (-1, 0) and B (0, 1)

(ii) and  $-x + y = 0$  passes through the point O (0, 0) and C (1, 1)

The line AB and OC are parallel. The region xoy plane which satisfy  $x - y \leq -1$  lies above the line AB and region xoy plane which satisfy  $x + y \leq 0$  lies below the line OC. Thus there is no feasible region. Hence, Max  $z$  does not exist



3. There is a factory located of each of the two places P and Q from thus location a certain commodity is delivered to each of thus depots situated at A, B and C. The weekly requirements of the depots are resp. 5,5 and 4 units of the commodity while the production capacity of the factories at P and Q are resp. 8 and 6 units the cost of transportation per unit is given below.

From \ To	Cost (in Rs.)		
	A	B	C
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depots in order that the transportation cost is minimum formulates the above LPP mathematically and then solve it.

**Solution :** Minimize  $z = x - 7y + 190$  subject to

$$x + y \leq 8$$

$$x + y \geq 4$$

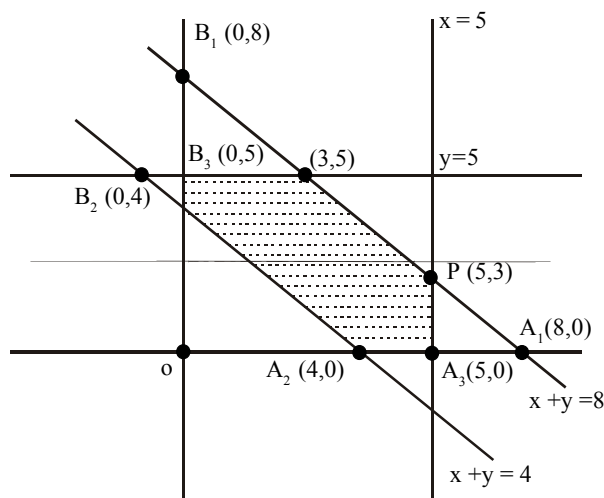
$$x \leq 5$$

$$y \leq 5$$

and  $x \geq 0, y \geq 0$

line  $x + y = 8$  gives  $B_1(0, 8)$  and  $A_1(8, 0)$ , line  $x + y = 4$  gives  $B_2(0, 4)$  and  $A_2(4, 0)$ ,  $x = 5$  and  $y = 5$  passes through the point  $A_3(5, 0)$  and  $B_3(0, 5)$ .

The coordinate of the corner points of the feasible region  $A_2 A_3 P Q B_3 B_2$  are  $A_2(4, 0)$ ,  $A_3(5, 0)$ ,  $P(5, 3)$ ,  $B_3(0, 5)$ ,  $Q(3, 5)$  and  $B_2(0, 4)$



$$\text{At, } A_2(4, 0) \Rightarrow z = 4 - 7 \times 0 + 190 = 194$$

$$\text{At, } A_3(5, 0) \Rightarrow z = 5 - 7 \times 0 + 190 = 195$$

$$\text{At, } P(5, 3) \Rightarrow z = 5 - 7 \times 3 + 190 = 174$$

$$\text{At, } Q(3, 5) \Rightarrow z = 3 - 7 \times 5 + 190 = 158$$

$$\text{At, } B_3(0, 5) \Rightarrow z = 0 - 7 \times 5 + 190 = 155$$

$$\text{At, } A_2(4, 0) \Rightarrow z = 4 - 7 \times 4 + 190 = 162$$

clearly  $z$  is minimum of  $x = 0, y = 5$  and minimum value of  $z$  is 155.

### PROBLEM FOR PRACTICE (SECTION-A)

1. Minimise  $z = x + 2y$  subject to

$$2x + y \geq 3, \quad x + 2y \leq 6, \quad \text{and } x \geq 0, \quad y \geq 0$$

2. Show that the minimum of  $z$  occurs at more than two points. Minimize and maximize  $z = 5x + 10y$  subject to  $x + 2y \leq 120, x + y \geq 60, x - 2y \geq 0, x, y \geq 0$

3. Maximise  $z = -x + 2y$  subject to

$$x \geq 3, \quad x + y \geq 5, \quad x + 2y \geq 6, \quad x, y \geq 0$$

4. Minimize and maximize  $z = 5x + 2y$  subject to

$$\begin{aligned} -2x - 3y &\leq -6 \\ x - 2y &\leq 2 \\ 3x + 2y &\leq 12 \\ -3x + 2y &\leq 3 \\ x, y &\geq 0 \end{aligned}$$

5. Maximize and minimize  $z = 3x + 5y$

$$3x - 4y + 12 \geq 0, \quad 2x - y + 2 \geq 0, \quad 2x + 3y - 12 \geq 0, \quad 0 \leq x \leq 4, \quad y \geq 2$$

6. Maximize  $z = 3x_1 + 5x_2$  subject to

$$x_1 + 3x_2 \geq 3, \quad x_1 + x_2 \geq 2, \quad x_1, x_2 \geq 0$$

## SECTION - B

1. A manufacturer produces two types of steel trunks. He has two machines A and B. The first type of the trunk requires 3 hours on machine A and 3 hours on machine B. The second type of trunk requires 3 hours on machine A and 2 hours on machine B. Machine A and B are run daily for 18 hours and 15 hours respectively. There is a profit of Rs. 30 on the first type of the trunk and Rs. 25 on the second type of the trunk. How many trunk of each type should be produced and sold to make maximum profit?
2. A young man drives his vehicle at 25 km/hr. he has to spend Rs. 2 km on petrol. If he drives it at a faster speed of 40 km/hr. the petrol cost increase to 5 km. he has to spend Rs. 100 on petrol and travel within one hour. Express this as an LPP and solve the same.
3. A gardener has a supply of fertilizer of type I which consist of 10% nitrogen and 6% phosphoric acid and type II fertilizer which consist of 5% nitrogen and 10% phosphoric acid. After testing the soil condition he finds that he needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If the type I fertilizer cost 60 paise per kg and type II fertilizer cost 40 paise per kg. determine how many kg of each fertilizer should be used so that nutrient requirement are met of minimum cost. What is the minimum cost.
4. A dietician has to develop a special diet using two foods P and Q each packet (containing 30g) of food P contains 12 unit of calcium, 4 unit of iron, 6 unit of cholesterol and 6 unit of vitamin A. Each packet of the same quantity of food Q contain 3 unit of calcium, 20 unit of iron, 4 unit of cholesterol and 3 unit of vitamin A. A diet requires at least 240 unit of calcium, at least 460 unit of iron and at most 300 unit of cholesterol. How many packet of each food should be used to minimize the amount of vitamin A in diet? What is the minimum amount of vitamin A?
5. An Aeroplane can carry maximum of 200 passengers. A profit of Rs. 1000 is made on each executive class ticket and profit of Rs 600 is made on each economy class ticket. The airline reserve at least 20 seat for executive class. However, at least 4 times as many passenger prefer to travels by economy class than by executive class. Determine how many tickets of each type be sold in order to maximise the profit for the airline. What is maximum profit.
6. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops D, E and F whose requirement are 60, 50, 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given below. How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

Transportation Cost per quintal (in Rs.)		
From \ To	A	B
D	7	3
E	6	4
F	3	2

## ANSWER (SECTION-A)

1. Minimum  $z = 6$  at all the point on the line joining the point  $(6, 0)$  and  $(0, 3)$ .
2. Minimum  $z = 300$  at  $(60, 0)$   
Maximum of  $z = 600$  at lline joining  $(120, 0)$  and  $(60, 30)$ .
3.  $z$  has no max value.
4. Minimum value  $\frac{63}{13}$  at  $(\frac{3}{13}, \frac{24}{13})$ .  
Maximum value of  $z$  is 19 at  $(\frac{7}{2}, \frac{3}{4})$ .
5. Minimum value of  $z = 19$  at  $(3, 2)$   
Maximum value of  $z$  is 42 at  $(4, 6)$
6. 7.

## SECTION B

1. 3 trunk of each type.
2. At  $25\text{km/hr} \rightarrow 50/3$  km,  
At  $40\text{km/hr} \rightarrow 40/3$  km.  
Max distance = 30km.
3. Type I  $\rightarrow 100$  kg, Type II  $\rightarrow 80$  kg  
Max Cost = Rs. 92.
4. Food P = 15 packet  
Food Q = 20 Pecket  
Minimum amount of vitamin A = 150 unit.
5. Executive class = 40 tickets.  
Economy class = 160 tickets  
Maximum profit = Rs. 1,36,000

## PROBABILITY

- \* Important definition and facts :-
- \* Conditional probability :- Conditional probability of occurence of event A, given that the event B has already occured and  $P(B) \neq 0$  is given by  $P(A/B) = \frac{P(A \cap B)}{P(B)}$  provided

$$P(B) \neq 0 \text{ and } P(B/A) = \frac{P(A \cap B)}{P(A)}, P(A) \neq 0$$

- \* Property :- 1.  $P(A'/B) = 1 - P(A/B)$     2.  $P(A \cup B/F) = P(A/F) + P(B/F) - P(A \cap B/F)$
- 3.  $P(A \cup B/F) = P(A/F) + P(B/F)$  when A and B are mutually Exclusive.

**Independent event :** Let E and F be the two events such that probability of occurrence of one of them is not affected by occurrence of the other. Then they are called independent event.

(a) Two event E and F are said top be independent if –

$$P\left(\frac{F}{E}\right) = P(F) \text{ provided } P(E) \neq 0$$

$$P\left(\frac{E}{F}\right) = P(E) \text{ provided } P(F) \neq 0$$

(b) Two events A and B are said to be independent if  $P(A \cap B) = P(A) \cdot P(B)$

\* **Baye's Thaorem :** If  $E_1, E_2, \dots, E_n$  are events which constitute portion of sample space i.e.  $E_1, E_2, \dots, E_n$  are pair wise disjoint and  $E_1 \cup E_2 \cup \dots \cup E_n = S$  and A be any event with non zero probability. Then

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{j=1}^n P(E_j)P\left(\frac{A}{E_j}\right)}$$

\* **Binomial distribution :** In expansion  $(q + p)^n$  General term  $P(r)$ , the probability of  $r$  success is given by  $P(r) = n_c q^{n-r} p^r$

\* Mean of random variable :- Let X be a random variable whase possible value  $x_1, x_2, \dots, x_n$  occure with probability  $p_1, p_2, \dots, p_n$  respectively. Then mean of X (denoted by  $\mu$ ) is

$$\sum_{i=1}^n x_i p_i$$

\* Variance of random variable :- Let X br the random variable whase possible values  $x_1, x_2, x_3, \dots, x_n$  occure with probanility  $p_1(x_1), p_2(x_2), \dots, p_n(x_n)$  resp. Then variance of X denoted by  $\text{var}(X)$  or  $\sigma_x^2$  is defined as  $\sigma_x^2 = \text{var}(X) = \sum_{i=1}^n (x_i - \mu)^2 p(x_i)$  and standered deviation

$$\sigma_x = \sqrt{\sum_{i=1}^n (x_i - \mu)^2 p(x_i) p_i}$$

**Note :** (a) Mean =  $np$

(b) Variance =  $npq$

(c) S.D =  $\sqrt{npq}$

**Illustrative Example :**

1. In a hostel 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspaper. A student is selected at random.

- (i) Find the probability that she read neither Hindi nor English news paper
- (ii) If she read Hindi newspaper, find the probability that she read English news paper.
- (iii) If she reads English newspaper find the probability that she read Hindi newspaper.

**Solution :**

Hindi : Student reads Hindi newspaper

English : Student reads English newspaper

$$\therefore P(H) = \frac{60}{100} = \frac{3}{5}, \quad P(E) = \frac{40}{100} = \frac{2}{5}, \quad P(H \cap E) = \frac{20}{100} = \frac{1}{5}$$

$$\begin{aligned} \text{(i) Required probability} &= P(\bar{H} \cap \bar{E}) = P(\overline{H \cup E}) = 1 - P(H \cup E) \\ &= 1 - \{P(H) + P(E) - P(H \cap E)\} \\ &= 1 - \left\{ \frac{3}{5} + \frac{2}{5} - \frac{1}{5} \right\} = 1 - \frac{4}{5} = \frac{1}{5} \end{aligned}$$

$$\text{(ii) Required Probability} = P\left(\frac{E}{H}\right) = \frac{P(H \cap E)}{P(H)} = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3}$$

$$\text{(iii) Required Probability} = P\left(\frac{H}{E}\right) = \frac{P(H \cap E)}{P(E)} = \frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{2}$$

2. A man speaks truth 4 out of 5 times. He throw a dice and report that it is Six what a Probability that it is actually Six.

**Solution :**  $E_1$  : the event that man speak truth

$E_2$  : the event that man not speak truth

Let A be event that the man reports Six

$$\therefore P(E_1) = \frac{4}{5}, \quad P(E_2) = \frac{1}{5}, \quad P\left(\frac{A}{E_1}\right) = \frac{1}{6}, \quad P\left(\frac{A}{E_2}\right) = \frac{5}{6}$$

Probability that Six has occurred when man speaks that there is Six.

$$\begin{aligned} P\left(\frac{E_1}{A}\right) &= \frac{P(E_1) \cdot P\left(\frac{A}{E_1}\right)}{P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right)} \\ &= \frac{\frac{4}{5} \times \frac{1}{6}}{\frac{4}{5} \times \frac{1}{6} + \frac{1}{5} \times \frac{5}{6}} = \frac{4}{9} \end{aligned}$$

3. The sum and product of mean and variance of a Binomial distribution are 24 and 28 resp. find the distribution  $\equiv$

**Solution :** Mean  $np$  and var  $= npq$

$$\therefore \frac{A}{q} \quad np + npq = 24 \text{ and } np \times npq = 128$$

$$np(1+q) = 24 \text{ and } n^2 p^2 \times npq = 128$$

$$np = \frac{24}{1+q} \text{ and } n^2 p^2 = \frac{128}{q}$$

$$\left( \frac{24}{1+q} \right)^2 = \frac{128}{q} \Rightarrow 576q = 128(1+q)^2$$

$$\therefore 2q^2 - 5q + 2 = 0 \Rightarrow q = \frac{1}{2} \quad [\because q \neq 2]$$

$$\therefore p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\therefore n = 32$$

thus Binomial distribution is  $P(x=r) = {}^{32}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{32-r}, r = 0, 1, 2, \dots, 32$

4. A box contain 100 bulbs out of which 20 we defective. 10 bulbs are selected at random find the probability that (i) All are defective (ii) none is defective (iii) at least one is defective (iv) At most one is defective.

**Solution :**  $n = 10$

$$\text{Probability of getting defective bulb } P = \frac{20}{100}$$

$$\text{Probability of getting not defective bulbs } q = \frac{80}{100}$$

$$\text{New } P(r) = {}^{n}C_r (P)^r (q)^{n-r}$$

$$(a) \quad v = 10 \quad \therefore p(10) = {}^{10}C_{10} \left(\frac{1}{5}\right)^{10} \left(\frac{4}{5}\right)^{10-10} = \left(\frac{1}{5}\right)^{10}$$

$$(b) \quad v = 0 \Rightarrow p(0) = {}^{10}C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10-0} = \left(\frac{4}{5}\right)^{10}$$

$$(c) \quad v \geq 1 \quad \therefore p(r \geq 1) = 1 - p(0) = 1 - \left(\frac{4}{5}\right)^{10}$$

$$\begin{aligned}
 \text{(d) } v \leq 2 \quad \therefore p(r \leq 2) &= p(0) + p(1) + p(2) \\
 &= 10c_0 \left(\frac{1}{5}\right)^{10-0} \left(\frac{4}{5}\right)^0 + 10c_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{10-1} + 10c_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{10-2} \\
 &= \frac{101}{25} \left(\frac{4}{5}\right)^8
 \end{aligned}$$

5. Two person A and B throw a coin alternately till one of them gets head and wins the game. Find their respective chance of wining.

**Solution :**

$p(A) = \frac{1}{2}$ ,  $p(\bar{A}) = \frac{1}{2}$ ,  $p(B) = \frac{1}{2}$ ,  $p(\bar{B}) = \frac{1}{2}$  A wins the game if he throw coin in 1<sup>st</sup>, 3<sup>rd</sup>, 5<sup>th</sup>.....

$\therefore$  Probability of A winning the game is

$$p(A) + p(\bar{A})p(\bar{B})p(A) + \dots$$

$$\frac{1}{2} + \left(\frac{1}{2}\right)^3 + \dots$$

$$\therefore \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$$

$$\therefore \text{Chance of B winning} = 1 - \frac{2}{3} = \frac{1}{3}$$

6. Find the mean and variance of number obtained on a throw of an unbiased die.

**Solution :**  $S = \{1, 2, 3, 4, 5, 6, \}$

$$\therefore P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

<b>X</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>P(x)</b>	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\therefore \text{Mean} = E(x) = 1 + \frac{1}{6} + 2 + \frac{1}{6} + 3 + \frac{1}{6} + 4 + \frac{1}{6} + 5 + \frac{1}{6} + 6 + \frac{1}{6} = \frac{21}{6}$$

$$\text{Also } = E(x) = 1^2 + \frac{1}{6} + 2^2 + \frac{1}{6} + 3^2 + \frac{1}{6} + 4^2 + \frac{1}{6} + 5^2 + \frac{1}{6} + 6^2 + \frac{1}{6} = \frac{91}{6}$$

$$\therefore \text{var}(x) = E(x^2) - [E(x)]^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{35}{12}$$

## PRACTICE EXERCISE

1. A problem in maths is given to 3 students whose chance of solving it are  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ . What is the probability that the problem is solved.
2. If  $P(A) = \frac{1}{2}$ ,  $P(B) = x$  and  $P(A \cup B) = \frac{3}{5}$  find  $x$  if they are mutually exclusive.
3. If  $2P(A) = P(B) = \frac{5}{13}$  and  $P\left(\frac{A}{B}\right) = \frac{2}{5}$  find  $P(A \cup B)$
4. Probability of happening of three independent events are  $P_1, P_2, P_3$ . What is the probability that at least one of the events will happen.
5. If  $P(A) = \frac{1}{2}$ ,  $P(A \cup B) = \frac{3}{5}$ ,  $P(B) = P$  then find  $P$  if they are independent.
6. If  $A$  and  $B$  are two events such that  $P(A) = 0.3 = P(B) = 0.6$  and  $P(B/A) = 0.5$  find  $P(A/B)$
7. If  $P(A) = \frac{3}{8}$ ,  $P(B) = \frac{1}{2}$  and  $P(A \cap B) = \frac{1}{4}$  find  $P\left(\frac{\bar{A}}{B}\right)$ .
8. A fair die is rolled consider the event  $A = \{1, 3, 5\}$ ,  $B = \{2, 3\}$  and  $C = \{2, 3, 4, 5\}$  find  $P = (A \cup B / C)$
9. One card is drawn from a pack of 52 cards so that each card is equally likely. Then prove that  
 $A$ : The Card drawn is spade.  
 $B$ : The Card drawn is an ace  
are independent.
10. Event  $A$  and  $B$  such that  $P(A) = \frac{1}{2}$ ,  $P(B) = \frac{7}{12}$  and  $P(\text{not } A \text{ or not } B) = \frac{1}{4}$  state whether  $A$  and  $B$  are independent.
11.  $P(A) = \frac{1}{4}$ ,  $P(B) = \frac{1}{2}$ ,  $P(A \cup B) = \frac{1}{8}$  find  $P(\text{neither } A \text{ nor } B)$

## SECTION -B

1. Probability of solving a problem independently by  $A$  and  $B$   $\frac{1}{2}$  and  $\frac{1}{3}$  respectively. If both try to solve the problem independently. Find the probability that (i) problem is solved (ii) exactly one of them solve the problem.
2. A card from pack of 52 cards is lost from the remaining cards of the pack, two cards are drawn and are found to be both diamond. Find the probability of the lost card being a diamond.
3. If the sum of the mean and variance of binomial distribution for 5 trials is 1.8 find the distribution.
4. Two cards are drawn at random one by one without replacement from a well shuffled pack of 52 cards. Find the probability distribution of number of aces.
5. Two cards are drawn successfully with replacement from pack of 52 cards. Find the probability distribution of kings.

6. A pair of dice is thrown 7 times. If getting a total of 7 is considered a success. What is the probability of
  - (i) no success, (ii) 6 success, (iii) at least 6 success, (iv) At most 6 success.
7. Of people having HIV, 90% of the test detect the disease but 10% go undetected of people free of HIV 99% of the test are judged HIV but 1% are diagnosed as showing HIV +ve from a large population of which 0.1% having HIV one person is selected at random given the HIV test and the pathologist reports him/her as HIV +ve. What is the probability that the person actually has HIV.
8. An urn contains 5 red and black balls. A ball is drawn at random and its colour is noted and is returned to the urn. Moreover 2 additional balls of the colour are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red.
9. A letter is known to have come either from TATANAGAR or CALCUTTA on the envelope just two consecutive letters TA are visible. What is the probability that the letter has come from (i) Calcutta (ii) Tatanagar.
10. Three persons A, B, C throw a die in succession till one gets a 'six' and wins the game. Find their respective chance of winning if A begins.
11. A husband and his wife appear for an interview for two posts. The probability of the husband's selection is  $\frac{1}{7}$  and that of wife's selection is  $\frac{1}{5}$  what is the probability that
  - (i) both of them will be selected.
  - (ii) only one of them will be selected.
  - (iii) at least one of them will be selected.
  - (iv) None of them will be selected.

### SECTION - C

1. Five cards are drawn successively with replacement from a well shuffled pack of 52 cards what is probability that
  - (i) All the five cards are spade.
  - (ii) Only three cards are spade.
  - (iii) None is a spade.
2. A and B throw alternately a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. Find their respective chance of winning if A begins.
3. A coin is biased so that the head is 3 times as likely to as tails. If the coin is tossed twice. Find the probability distribution of No. of tails. Also find the mean and variance.
4. Two cards are drawn simultaneously without replacement from a pack of 52 cards. Compute the mean, variance and standard deviation for the No. of kings.
5. A random variable  $x$  has the following probability distribution values up to  $x$ .
 

$x :$	0	1	2	3	4	5	6	7
$P(x) :$	0	$k$	$2k$	$2k$	$3k$	$k^2$	$2k^2$	$7k^2+k$

 Find each of the following.
  - (i)  $k$  (ii)  $P(x < 6)$  (iii)  $P(x \geq 6)$  (iv)  $P(0 < x < 5)$

6. A and B be two independent events the probability of their simultaneous occurrence is  $\frac{1}{8}$  and the probability that neither occurs is  $\frac{3}{8}$  find P(A) and P(B).

**Answer (Section -A)**

1.  $\frac{3}{4}$       2.  $\frac{1}{10}$       3.  $\frac{11}{26}$       4.  $1 - (1 - P_1)(1 - P_2)(1 - P_3)$   
 5.  $\frac{1}{5}$       6.  $\frac{1}{4}$       7.  $\frac{3}{4}$       8.  $\frac{3}{4}$       11.  $\frac{4}{8}$

**Section - B**

1.  $\frac{2}{3}, \frac{1}{2}$       2.  $\frac{11}{50}$       3.  $5_{c\gamma}(0.2)^2 (0.8)^{5-\gamma}$       4.  $\frac{188}{221}, \frac{32}{221}, \frac{1}{221}$   
 5.  $\frac{144}{169}, \frac{24}{169}, \frac{1}{169}$       6.  $(\frac{5}{6})^7, 35(\frac{1}{6})^7, (\frac{1}{6})^5, 1 - (\frac{1}{6})^7$   
 7.  $\frac{90}{1089}$       8.  $\frac{1}{2}$       9.  $\frac{4}{11}, \frac{7}{11}$       10.  $\frac{36}{91}, \frac{30}{91}, \frac{25}{91}$   
 11.  $\frac{1}{35}, \frac{2}{7}, \frac{11}{35}, \frac{24}{35}$

**Section - C**

1.  $\frac{1}{1024}, \frac{45}{512}, \frac{243}{1024}$       2.  $\frac{30}{61}, \frac{31}{61}$   
 3.  $\frac{9}{16}, \frac{6}{16}, \frac{1}{16}$  and mean =  $\frac{3}{2}$ , Var.  $\frac{3}{8}$       4.  $\frac{34}{321}, \frac{6800}{221}, 0.37$   
 5.  $\frac{1}{10}, \frac{81}{100}, \frac{19}{100}, \frac{4}{5}$   
 6.  $P(A) = \frac{1}{2}$  and  $P(B) = \frac{1}{4}$  or,  $P(B) = \frac{1}{2}$  and  $P(A) = \frac{1}{4}$