STUDY MATERIAL

KENDRIYA VIDYALAYA SANGATHAN

GUWAHATI REGION

CLASS–XII

MATHEMATICS

(For Academic Session : 2008–2009)

— *Patron* —

Sh. U.N. Khawarey
Assistant Commissioner
KVS (RO), Guwahati

— *Editor* —

Sh. Anil Kumar
Principal
KV, ONGC, Jorhat

— *Contributors* —

1. Dr. A.N. Mishra
   PGT (Maths)
   KV, ONGC, Jorhat
2. Sh. D.K. Jha
   PGT (Maths)
   KV, CRPF, Amerigog
3. Sh. T. Narsimha
   PGT (Maths)
   KV, Happy Valley
4. Shri R.K. Mall
   PGT (Maths)
   KV, Nagaon

CONTENTS

A Word for the Student

It gives us great pleasure in bringing out the study material according to the new syllabus of C.B.S.E. A group of experienced teachers and Principals have put in their hard work in preparing the booklet in tune with the guidelines of C.B.S.E with a view to provide you the synopsis of the whole course at a glance.

The study materials have been especially designed to help in revision and dummy examination. Use these materials to their fullest besides, using other materials. The study material is ready to give you command over concept and ability to answer any question in its proper way.

It has been told "It is not the feathers but the spirit that flies." It is you only who is going to shape your future. Put in your best endeavor keeping in view of the following :-

- *a. Fix a target for yourself and plan strategies to achieve these with the help of teachers & parents.*
- *b. Make a time- table for regular studies or revision of minimum 6-8 hrs daily. Do not study at a stretch, take a few minutes break.*
- *c. Writing practice should be done by solving different questions.*
- *d. Never read mathematics. Always adopt pen & paper while you are with mathematics.*
- *e. As far as possible continue with the normal routine of sleeping & eating.*
- *f. A balanced diet will boost energy.*
- *g. Improve you concentration by relaxation practice deep breathing or yoga. Regular and moderate exercises reduce stress by relaxing tensed muscles.*
- *h. Quickest and most effective way of eliminating stress is to shut down your eyes and take deep breaths.*
- *i. Believe in yourself and prepare well. Take help of teachers and parents from time to time and say "I do not believe in things to happen but I believe to make them happen."*

Above all, never fear exams, avoid panic and most important do not worry about vour results. Put in your best efforts and good result shall be yours

Wish you all the best

Yours

(U. N. Khawarey)
Asstt. Commissioner
KVS, GR, Guwahati

A WORD TO THE PARENTS

Dear parents,

Ritcher says, 'Every human being has two educations-that which is given to him/her, and the other, that he gives to himself. Of the two kinds, the later is by far the most valuable. Felton adds, "Knowledge will not be acquired without pains and application. It is troublesome and deep digging for pure water; but once you come to the spring, they rise up & meet you. "Batista concludes, "The best inheritance that a parent can give to his children is a few minutes of his time everyday.

While thanking you for your trust in us by putting your wards under our care, I would like to recall you that this is the high time for us to be supportive to our students as teachers as well as parent. As a parent you may take care of the following in letter and spirit in respect of your ward:

1. *Please watch the food habits of your children, green vegetable, milk honey, egg, fish, hygienic and fibrous food is the need of day for your children.*
2. *They must have sometime (say 60 minutes) for physical activities (be it games & sports, running, swimming Yoga or other Exercise, etc.) They should have some time (say 60-90 minutes) for fun, not necessarily in one spell. Study hours should have breaks after every 60-90 minutes.*
3. *It is not only important how much he /she studies but also well he/she studies.*
4. *Please always encourage your children. Please do not condemn them even if they do not perform up to your expectation in an exam. Make them aware of the mis takes to ensure that the same are not repeated in future.*
5. *Please don't compare him / her with any other child. Compare him / her in terms of Time. As long as he/she is improving, there is no reason for disappointment.*
6. *Please don't criticize the school or any teacher infront of the child. 'This will lower his/ her faith in the system.*
7. *Boost his/her morale and let him/her have confidence in himself/herself.*
8. *Help him/her indirectly by facilitating but for God's sake don't DO HIS/HER WORK since he/she has to do his/her own work and that he/she will learn only by doing.*
9. *Help him/her to be able to think in a systemic and coherent manner.*
10. *Encourage your child for discussion, mental exercise and experimentation.*
11. *Please ensure that atmosphere at home is congenial for study and practice.*

Let us all have faith in him/her and let us help him/her achieve to his/her fullest potential. With Best Wishes.

Your,

(U.N.KHAWAREY)
Asstt. Commissioner
KVS, GR, Guwahati

A WORD FOR THE TEACHER

Dear Teachers,

You must be happy to get the study materials for your students. I am aware that the materials are fruits of your hard work, dedication and love for your students. By now, you must have made the following exercises:

1. *Checking status of your students subject-wise, area-wise and competence-wise.*
2. *You must have identified your low achiever and high-achiever students and also the areas which require your attention in terms of priority.*
3. *You must have set different targets for the low and high achievers.*
4. *You must have prepared plans for higher and additional tasks and motivation for the bright students.*
5. *You must have earmarked tasks for the low achievers followed by expanded coverage areas.*

I would like to advise you to use the contents of the Study Materials to its fullest by

i) *Constant monitoring.*
ii) *Emphasizing on concept and not on learning by 'Not '.*
iii) *Emphasizing on Surprise Element and not on guess work.*
iv) *Using thinking as a tool by encouraging the students to make reference, analysis, summaries etc.*
v) *Motivating the students to self-study, self test and self-evaluation.*
vi) *Ensuring that right type of study Techniques are adopted by students.*

I would also like to add that no student should be humiliated and insulted. Please tell him/her first his/her plus points-strengths followed by his/her weak points. However, all the weak areas should not be pointed out in one go.

I hope you would use your talent and experiences by giving personal attention to each student and by giving them dead-line-based time-frame for learning, noting, understanding and recalling the understood contents as well as by having more and more practice.

I wish that your efforts will bring about a quantitative as well as qualitative upsurge in the result of your subject in particular and school in general.

Thanking you.

Your's faithfully,

(U.N. KHAWAREY)
ASSISTANT COMMISSIONER

Editorial

***"The beautiful thing about
Education is that, no one
Can take it away from you"***

***"There is a brilliant child,
Locked inside every student"***

***"What I hear I forget,
What I see I remember,
What I do I understand."***

The purpose of preparing study material in Mathematics for class XII is to provide strategy for better performance in the CBSE Board Examination 2009. This material gives basic concepts, key notes, quick reference to the students. It is more application oriented and covers all types of questions. The problems are given in three groups as per the design of a questio. Some sample answers have also been provided to guide the students in answering.

Some model question papers along with the some set's solution with marking scheme have been provided. The question papers have been prepared by keeping the marks distribution as per the new design. All efforts have been taken to fulfill the requirement of an average student. Mathematics is a subject which requires a lot of practices so one has to do accordingly.

I express my sincere thanks to all the teachers who contributed a lot and prepared this study material. I express my thanks to Shri R.J. Ram, Principal, K.V. Nagaon, Shri R.P. Dwivedi, Principal, K.V. Happy vallely & Principal, KV, CRPF, Amengog, who helped a lot in preparing this material. I believe this study material will be very useful to the students and serve the purpose.

"What man want is not to list, it is a purpose not the power to achieve but the will to hard work." *As proverb goes,*

***"If it is to be
It is up to me."***

(Anil Kumar)
Principal
KV, ONGC, Jorhat

CHAPTERS WITH WEIGHTAGE

(For Acamedic Session – 2008—2009)
CLASS—XII
Subject : Mathematics

ONE PAPER THREE HOURS MARKS : 100

UNITS	CHAPTERS	MARKS
I	RELATION AND FUNCTIONS 1. Relations and Functions 2. Inverse Trigonometric Functions	10
II	ALGEBRA 1. Matrices 2. Determinants	13
III	CALCULUS 1. Continuity and Differentiability 2. Applications of Derivatives 3. Integrals 4. Applications of Integrals 5. Differential Equations	44
IV	VECTORS AND THREE-DIMENSIONAL GEOMETRY 1. Vectors 2. Three-Dimensional Geometry	17
V	LINEAR PROGRAMMING 1. Linear Programming	06
VI	PROBABILITY 1. Probability	10
		100

PART–I

CHAPTER–1

Relations and Funtions

Key point

(I) A relation in a set A is a subset of A × A. Thus, R is a relation in a set $A \Leftrightarrow R \subseteq A \times A$.

Types of relations

(a) Void/Empty Relation :

A relation R in a set A is called empty relation if no element of A is related to any element of A i.e R= $\varphi \subset A \times A$

(b) Universal relation :

A relation R in a set A is called universal relation if each element of A is related to every element of A i.e R= A×A

(c) Identity relation : The relation IA = $\{(x, x) : x \in A\}$ is called identity relation on set A.

Classification of Relations :

(i) Rellexive : A relation R in a set is said to be rellexive if $(a,a) \in R$ for all $a \in A$.

(ii) Symmetric : A relation R in a set A is said to be symmetric if–

$$(a_1, a_2) \in R \Rightarrow (a_2, a_1) \in R \forall a_1, a_2 \in A$$

(iii) Transitive :A relation R in a set A is said to be transitive if

$$(a_1, a_2) \in R \text{ and } (a_2, a_3) \in R \Rightarrow (a_1, a_3) \in R \forall a_1 a_{21} a_3 \in A$$

(iv) Equivalence relation :A relation R in a set A is said to be equivalence relation if it is reflexive, symmetric and transitive.

Function :

If *A* and *B* are two sets then a rule *f*, under which to every element *x* of the set *A* there corresponds one and only element *y* of set *B* is called the function from *A to B*.

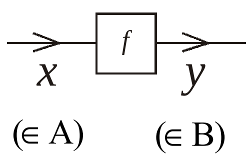
If a pre-image is denoted by *x* and all image is denoted by *y*, then we can write

$y = f(x)$

Image ($\in$ B) Pre-image ($\in$ A)

f(*x*) is called the value of the function *f* at the point *x*.

Machine Model of Function



A function $f : A \rightarrow B$ is called real valued if the image of every element of A under f is a real number.

i.e. if $f(x) \in R \ \forall \ x \in A$

$$y = f(x)$$

$\downarrow$ Dependent Variable $\quad \downarrow$ Independent Variable

Various Types of functions :

1. One-one (Injective) function :

If $f : A \rightarrow B$ then f is called one-one if no element of B is an image of more than one element of A.

Symbolically, $\quad x \neq y \Rightarrow f(x) \neq f(y)$

Above condition is equivalent to

$$f(x) = f(y) \Rightarrow x = y$$

Functions which are not one-one are called many one.

2. Onto (Subjective) Functions :

Let $f : A \rightarrow B$ then f is onto if every element of B is the f-image of at least one element of A.

Here, $\quad R_f = B$

(Range) = (Codomain)

If there exists at least one element in B which is not f-image of any element of A then f is called an into function.

Here, $\quad R_f \subset B$

(Range) (Codomain)

"A function which is both injective and surjective is called bijective."

Composition of Functions :

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two given functions. Then, the composition of f and g denoted by *gof* is the function, defined by –

$(gof) : A \rightarrow C : (gof)(x) = g(f(x)) \ \forall \ x \in A$

Clearly, dom (*gof*) = *dom (f)*

Invertible Function :

Let $f : A \rightarrow B$. If there exists a function $g : B \rightarrow A$ such that $gof = I_A$ and $fog = I_B$ then f is called invertible function and g is called the inverse of f. We write $f^{-1} = g$. Inverse of a function is defined if the function is one-one and onto.

Binary Operation :

A function f : A $\rightarrow$ B, where A is a set, is considered as a unitary operation in the sense that an element of A is associated to each singleton sub set $\{a\}$ of A.

If an element of A is associated uniquely with each subset of two element of A (the order of the element being taken account), we obtain a binary operation on set A.

In other words, a map A × A $\rightarrow$ B is called binary operation in A and if B Ì A, then A is said to be closed with reference to the operation.

i Addition of two real numbers is a binary operation.

ii Multiplication is also binary operation on R, while division is not binary operation on R, because division by 0 is not defened. But division is a binary operation on R – $\{0\}$.

Properties of Binary Operations :

i. **Commutative :** If a $*$ b = b $*$ a $\forall$ $a_1 b \in A$ then $*$ is commutative.

ii. **Associative :** If a $*$ (b $*$ c) = (a $*$ b) $*$ c $a, b, c \in A$ then $*$ is associative.

iii. **Existence of Identity Element :** If $*$ is binary operation on A and there is $e \in A$ such that a $*$ e = a = e $*$ a, then e is the identity element.

iv. **Existence of Inverse Element :** An element b in a set A is said to be inverse of an element $a \in A$ with reference to binary operation $*$ if a $*$ b = e = b $*$ a

ILLUSTRATIVE EXAMPLES

Example-I : Let A be the set of all line in xy – plane and let R be a relation in A, defined by

$$R = \left\{(l_1,\ l_2) : l_1 \parallel l_2\right\}$$

Show that R is an equivalence relation in A.

Solution :

(i) Reflexivity :

Let L be an arbitary line in A.

Then, $L \parallel L \Rightarrow (L_1 L) \in R \quad \forall\ L \in A$

Thus R is reflexive

(ii) Symmetry :

Let $L_1, L_2 \in A$ such that $(L_1, L_2) \in R$, then $(L_1, L_2) \in R \Rightarrow L_1 \parallel L_2$

$\Rightarrow L_2 \parallel L_1$

$\Rightarrow (L_2, L_1) \in R$

$\therefore$ R is symmetric

(iii) **Transitivity :** Let $L_1, L_2, L_3 \in A$ such that $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$

Then $(L_1, L_2) \in R$ and (L_2, L_3)

$\Rightarrow L_1 \parallel L_2$ and $L_2 \parallel L_3$

$\Rightarrow L_1 \parallel L_3$

$\Rightarrow (L_1, L_3) \in R$

$\therefore$ R is transitive.

Thus R is reflexive, symmetric and transitive.

Hence R is an equivalent relation.

Example-2 : Show that the function $f : R \to R : f(x) = 3 - 4x$ is one-one onto and hence bijective.

Solution : We have $f(x_1) = f(x_2)$

$\Rightarrow 3 - 4x_1 = 3 - 43x_2$

$\Rightarrow -4x_1 = -4x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

Now let $y = 3 - 4x$

$\Rightarrow x = \frac{3-y}{4}$

Thus for each $y \in R$ (Codomain of f), there exist $x = \frac{3-y}{4} \in R$ such that

$f(x) = f\left(\frac{3-y}{4}\right) = \left\{3 - 4 \bullet \frac{3-y}{4}\right\} = 3 - (3 - y) = y$

This shows that every element in co-domain of f has its Pre-image in domain f

$\therefore$ f is onto

Hence, the given function is a bijection.

Example- 3 : If $f(x) = e^x$ and $g(x) = \log x$ (x >o), show that *fog* = *gof*

Solution :

We have $f(x) = e^x$ and $g(x) = logx$, $(fog)(x) = f(g(x)) = f(logx) = e^{\log x} = x$ and $(gof)(x)$

$= g(f(x)) = g(e^x) = log\, e^x = xloge = x \times 1 = x$

Hence *fog* = *gof*.

Example- 4 : Let f : [-1,1] $\rightarrow$ Y : $f(x) = \frac{x}{x+2}, x \neq -2$ and Y = Range (f) : show that f is invertible and find f^{-1}.

Solution : We have $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1}{x_1+2} = \frac{x_2}{x_2+2}$$
$$\Rightarrow x_1x_2 + 2x_1 = x_1x_2 + 2x_2$$
$$\Rightarrow 2x_1 = 2x_2$$
$$\Rightarrow x_1 = x_2$$

$\therefore$ f is one.one

Since $R_f =$ y, so f is on to.

Thus f is one.one on to.

$\therefore$ f is inverlible.

Let y$\in$ Y. Then, there exist x$\in$ [-1,1] such that f(x) = y

Now, y = f(x)

$$\Rightarrow y = \frac{x}{x+2}$$
$$\Rightarrow x = \frac{2y}{1-y}$$
$$\Rightarrow f^{-1}(y) = \frac{2y}{1-y}$$

Thus, we define

$$f^{-1} : [-1,1] \rightarrow y : f^{-1}(y) = \frac{2y}{1-y}, y \neq 1$$

Example-5: In the set of natural numbers, whether the operation 'O' defined by mon = g.c.d (m,n) : m,n $\in$ N is (i) Commutative (ii) associative

Solution :

We have mon = g.c.d (m,n), m,n$\in$ N

(i) Since g.c.d (m.n) = g.c.d.(n,m)

$\therefore$ mon = nom

Thus O is commutative binary operation

(ii) Let l,m,n$\in$ N

then g.c.d.{l. g.c.d.(m,n)} = g.c.d (g.c.d(l,m),n), Þ l o (mon) = (lom) o n

$\therefore$ O is associative binary operation.

PROBLEMS

GROUP - A (1 MARK QUESTIONS)

Q1. Let A={1,2,3,4,6} and R be relation A defined by R={(a,b) : a$\in$ A, b$\in$ A and a divides b} Find (I) R (II) dom(R) (III) Range(R)

Q2. Show that the relation R = {(a,b) : a > b} on N is transitive but neither reflexive nor symmetric.

Q3. Let A = {1,2,3} and R= {(1,1),(2,2),(3,3),(1,2),(2,3)} show that R is reflexive but neither symmetric nor transitive.

Q4. Let $f : N \to N : f(x) = 2x + 4 \forall x \in N$. Show that f is one.one.

Q5. Find the domain of the real function, defined by

$$f(x) = \frac{1}{1-x^2}$$

Q6. Let R be the set of all real numbers. Let $f : R \to R : f(x) = \sin x$ and $g : R \to R : g(x) = x^2$. Prove that $gof \neq fog$.

Q7. Show that the function f : R$\to$R : f(x) = 2x + 3 is invertible and find f^{-1}

Q8. If $f(x) = \frac{5x+3}{4x-5}, (x \neq \frac{5}{4})$, show that f(f(x)) is an identity function.

Q9. Let f be the exponential function and g be the logarithmic function. What is (*fog*) (1)?

Q10. If $f(x) = \sqrt{x}$ $(x > o)$ and $g(x) = x^2 - 1$, Is *fog* = *gof* ?

GROUP - B (4 MARKS)

Q11. If R is a relation in N × N, show that the relation R defined by (a, b) R (c, d) if and only if ab =bc is an equivelence relation.

Q12. Let A be the set of all triangle in a plane and let R be a relation in A, defined by

$$R = \{(\triangle_1, \triangle_2) : \triangle_1 \cong \triangle_2\}$$

Show that R is an equivelence relation in A.

Q13. Give an example of a relation which is

(i) reflexive and transitive but not symmetric.

(ii) symmetric and transitive but not reflexive.

(iii) reflexive and symmetric but not transitive.

Q14. Define * on Z by a * b = a + b - ab. Show that * is a binary operation on Z which is commutative as well as associative.

Q15. Find the domain and range of the real function f, defined by $f(x) = \frac{x^2}{1+x^2}$

Q16. If $f : R \to R : f(x) = (3-x^3)^{\frac{1}{3}}$, show that (fof) (x) = x^3

Q17. Let $f : A \to B$ and $g : B \to A$ such that (gof) = I_A. Show that f is one-one and g is onto.

Q18. Let R_+ be the set of all non-negative real numbers. Let $f : R_+ \to [4, \alpha : f(x) = x^2 + 4]$. Show that f is invertible and find f^{-1}.

Q19. $f : w \to w : f(n) = \begin{cases} n-1, & when\ n\ is\ odd \\ n+1, & when\ n\ is\ even \end{cases}$

Show that f is invertible. Find f^{-1}.

Q20. Let $f : R \to R$ be defined by

$f(x) = x + |x|$

Determine whether or not f is onto.

Answers

Q. 1 : (i) R = {(1, 2), (1, 3), (1, 4), (1, 6), (2, 4), (2, 6), (3, 6)}

(ii) $dom(f) = \{1, 2, 3\}$

(iii) $Range(R) = \{2, 3, 4, 6\}$

Q. 5 : $Dom(f) = R - \{-1, 1\}$

Q. 7 : $f^{-1}(y) = \frac{1}{2}(y-3)$

Q. 9 : 1.

Q. 10 : No.

Q. 15 : $Dom\ (f) = R$

$Range\ (f) = \{y \in R\ ;\ 0 \le y < 1\}$

Q. 20 : Not Onto.

CHAPTER–2
Inverse Trigonometric Functions

Key point

Sl. No.	Function	Domain	Range (Principal Values)
1.	$y=\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
2.	$y=\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
3.	$y=\tan^{-1}x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
4.	$y=\cot^{-1}x$	R	$(0,\pi)$
5.	$y=\sec^{-1}x$	$R-[-1,1]$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$
6.	$y=\text{cosec}^{-1}x$	$R-(-1,1)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

1. (i) $\text{Sin}^{-1}\ (\text{Sinx})=\text{x},\ \text{x}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

   (ii) $\text{Cos}^{-1}(\text{Cosx})=\text{x}\in[0,\ \pi]$

   (iii) $\tan^{-1}\ (\text{tanx})=\text{x},\ \text{x}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

   (iv) $\text{Cot}^{-1}(\text{Cotx})=\text{x}\in[0,\ \pi]$

   (v) $\text{Sec}^{-1}\ (\text{Secx})=\text{x},\ \text{x}\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$

   (vi) $\text{Cosec}^{-1}\ (\text{Cosecx})=\text{x},\ \text{x}\in\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]-\{0\}$

2. (i) $\text{Sin}\ (\text{Sin}^{-1}\text{x})=\text{x},\ \ \text{x}\in[-1,\ 1]$

   (ii) $\text{Cos}\ (\text{Cos}^{-1}\text{x})=\text{x},\ \ \text{x}\in[-1,\ 1]$

   (iii) $\tan\ (\tan^{-1}\text{x})=\text{x},\ \ \text{x}\in \text{R}$

(iv) $\text{Cot}\left(\text{Cot}^{-1}x\right)=x,\ \ x\in R$

(v) $\text{Sec}\left(\text{Sec}^{-1}x\right)=x,\ \ x\in R-(-1,1)$

(vi) $\text{Cosec}\left(\text{Cosec}^{-1}x\right)=x,\ \ x\in R-(-1,1)$

3. (i) $\text{Sin}^{-1}\left(\frac{1}{x}\right)=\text{Cosec}^{-1}x \qquad (x\geq 1\ \text{or}\ x\leq -1)$

(ii) $\text{Cos}^{-1}\left(\frac{1}{x}\right)=\text{Sec}^{-1}x \qquad (x\geq 1\ \text{or}\ x\leq -1)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right)=\text{Cot}^{-1}x \qquad (x>0)$

4. (i) $\text{Sin}^{-1}(-x)=-\text{Sin}^{-1}x \qquad x\in[-1,\ 1]$

(ii) $\text{Cos}^{-1}(-x)=\pi-\text{Cos}^{-1}x,\quad x\in[-1,\ 1]$

(iii) $\tan^{-1}(-x)=-\tan^{-1}x,\quad x\in R$

(iv) $\text{Cot}^{-1}(-x)=\pi-\text{Cot}^{-1}(x),\quad x\in R$

(v) $\text{Sec}^{-1}(-x)=\pi-\text{Sec}^{-1}(x),\quad |x|\geq 1$

(vi) $\text{Cosec}^{-1}(-x)=-\text{Cosec}^{-1}(x),\quad |x|\geq 1$

5. (i) $\text{Cos}^{-1}x+\text{Sin}^{-1}x=\frac{\pi}{2},\ \ x\in[-1,1]$

(ii) $\tan^{-1}x+\text{Cot}^{-1}x=\frac{\pi}{2},\ \ x\in R$

(iii) $\text{Cosec}^{-1}x+\text{Sec}^{-1}x=\frac{\pi}{2},\ \ |x|\geq 1$

6. (i) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\ \ if\ \ xy<1$

(ii) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\ \ if\ \ xy>1$

(iii) $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\ \text{if}\ |x|<1$

7. (i) $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right),\ \ |x|<1$

(ii) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \ |x| \geq 0$

8. (i) $2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right] \ \frac{-1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$

(ii) $2\cos^{-1}x = \cos^{-1}\left(2x^2 - 1\right), \ \frac{1}{\sqrt{2}} \leq x \leq 1$

9. (i) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$

(ii) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left\{xy \mp \sqrt{1-x^2} \cdot \sqrt{1-y^2}\right\}$

10. $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^2\left(\frac{1-x^2}{1+x^2}\right)$

ILLUSTRATIVE EXAMPLES

Example 1 : Find the Principal value of each of the following :

(i) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (ii) $\operatorname{Cosec}^{-1}(-2)$ (iii) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution :

(i) We know that the Principal Value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \theta$

$\tan\theta = \frac{-1}{\sqrt{3}} = \tan\left(\frac{-\pi}{6}\right)$, where, $\frac{-\pi}{6} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Hence, the Principal value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is $\frac{-\pi}{6}$

(ii) We have that the Principal-Value branch of Cosex –1 is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Let $\operatorname{Cosec}^{-1}(-2) = \theta$

Then $\operatorname{Cosec}\theta = -2 = \operatorname{Cosec}\left(\frac{-\pi}{6}\right)$, Where $\frac{-\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Principal Value $= {}^{-\pi}/_{6}$

(iii) We have that the range of the Principal Value of Cos^{-1} is $[0, \pi]$.

Let $\quad \text{Cos}^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \theta$

$$\text{Cos}\theta = -\frac{1}{\sqrt{2}} = -\text{ Cos}\frac{\pi}{4} = \text{Cos}\left(\pi - \pi/4\right) = \text{Cos}\frac{3\pi}{4}$$

$$\therefore \frac{3\pi}{4} \in [0, \pi]$$

Principal Value $= \frac{3\pi}{4}$

Example 2 :

Prove that $\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\frac{1}{2}$

Solution :

Here $x = \frac{2}{11}, y = \frac{7}{24}$

So, $x.y = \frac{2}{11} \times \frac{7}{24} = \frac{7}{132} > 1$

$$\therefore \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\left(\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \times \frac{7}{24}}\right)$$

$$= \tan^{-1}\left(\frac{125}{264} \times \frac{264}{250}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

Example 3 : Prove that $\tan^{-1}\left(\frac{\cos x}{1 + \text{Sin}x}\right) = \frac{\pi}{4} - \frac{x}{2}$

Solution : We have

$$\text{L.H.S.} = \tan^{-1}\left(\frac{\cos x}{1 + \text{Sin}x}\right)$$

$$= \tan^{-1}\left\{\frac{\text{Sin}\left(\frac{\pi}{2} - x\right)}{1 + \text{Cos}\left(\frac{\pi}{2} - x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2.\text{Sin}\left(\frac{\pi}{4}-\frac{x}{2}\right).\text{Cos}\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2.\text{Cos}^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \frac{\pi}{4}-\frac{x}{2}$$

Example 4: It $\text{Cos}^{-1}x+\text{Cos}^{-1}y+\text{Cos}^{-1}z=\pi$ Prove that,

$$x^2+y^2+z^2+2xyz=1$$

Solution : We have

$$\text{Cos}^{-1}x+\text{Cos}^{-1}y+\text{Cos}^{-1}z=\pi$$

$$\Rightarrow \text{Cos}^{-1}x+\text{Cos}^{-1}y=\pi-\text{Cos}^{-1}z$$

$$\Rightarrow \text{Cos}^{-1}\left\{x.y-\sqrt{1-x^2}.\sqrt{1-y^2}\right\}=\pi-\text{Cos}^{-1}z$$

$$\Rightarrow \text{Cos}^{-1}\left\{xy-\sqrt{1-x^2}\,\sqrt{1-y^2}\right\}=\text{Cos}^{-1}(-z)$$

$$[\because \text{Cos}^{-1}(-x)=\pi-\text{Cos}^{-1}z]$$

$$\Rightarrow xy-\sqrt{1-x^2}.\sqrt{1-y^2}=-z$$

$$\Rightarrow xy+z=\sqrt{1-x^2}.\sqrt{1-y^2}$$

Squaring

$$x^2y^2 + z^2 + 2xyz = (1-x^2)(1-y^2)$$

Hence

$$x^2 + y^2 + z^2 + 2xyz = 1$$

INVERSE TRIGONOMETRIC FUNCTION
GROUP -A (1 MARKS)

Q.1. Write the domain and principle brance of $\tan^{-1}x$.

Q.2. Simplify, $\tan^{-1}\sqrt{\frac{1-\text{Cosx}}{1+\text{Cosx}}}, 0 \le x \le \pi$

Q.3. To simplify $\text{Cos}^{-1}(2x^2 - 1)$. What is the substitution for x.

Q.4. Write in simplest form—

$\text{Cos}^{-1}\left[4x^3 - 3x\right]$

Q.5. $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) =$ ________________

Q.6. Solve for x : $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x,\ x > 0$

Q.7. If $\sin^{-1}(1-x) - 2\sin^{-1}x = \pi/2$, then x is equal to (i) $0, \frac{1}{2}$ (ii) $1/2$ (iii) $1, 1/2$ (iv) 0

Q.8. Find the value of — Cot ($\tan^{-1}x + \text{Cot}^{-1}x$)

Q.9. Evaluate — $\sin\left[\pi/3 - \sin^{-1}\left(-1/2\right)\right]$

Q.10. Show that $\sin^{-1}\left\{\sin\frac{3\pi}{4}\right\} \ne \frac{3\pi}{4}$ and find its value.

Q.11. Solve for x, $\sin^{-1}\frac{2\alpha}{1+\alpha^2} + \sin^{-1}\frac{2\beta}{1+\beta} = 2\tan^{-1}x$

GROUP–A (ANSWERS) (1-Marks)

1. Ans. :- Domain : R, Range : $\left(-\pi/2,\ \pi/2\right)$

2. Ans. :- $\frac{x}{2}$

3. Ans. :- $x = \cos\theta$

4. Ans. :- $3\cos^{-1}x$

5. Ans. :- $\pi/4$

6. Ans. :- $1/\sqrt{3}$

7. Ans. :- (iv)
8. Ans. :- 0
9. Ans. :- 1
10. Ans. :- $\pi/4$
11. Ans. :- $\frac{\alpha+\beta}{1-\alpha\beta}$

GROUP–B (4 marks)

Q.1. Prove —

$$\tan^{-1}\left[\frac{3a^2x-x^3}{a\left(a^2-3x^2\right)}\right]=3\tan^{-1}\left(\frac{x}{a}\right)$$

Q.2. Prove —

$$\tan^{-1}\frac{m}{n}-\tan^{-1}\frac{m-n}{m+n}=\pi/4$$

Q.3. Prove —

$$4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\pi/4$$

Q.4. Write in simplest form —

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right),\ x<\pi$$

Q.5. Prove —

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right]$$

Q.6. If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$, then prove that —

$$9x^2-12xy\cos\theta+4y^2=36\sin^2\theta$$

Q.7. Solve for x —

$$\tan^{-1}(2x)+\tan^{-1}(3x)=\pi/4$$

Q.8. Solve for x —

$$\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\pi/4$$

Q.9. Write in simplest form —

$$\sin^{-1}\left[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right]$$

Q.10. Prove that —

$$\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x - a\sin x}\right) = \tan^{-1}\left(\frac{a}{b}\right) - x$$

Q.11. Solve —

$$\tan^{-1}(2+x) + \tan^{-1}(2-x) = \tan^{-1}\left(\frac{2}{3}\right)$$

GROUP–B (ANSWERS) (4 marks)

4. Ans. :- $\pi/4 - x$

7. Ans. :- $1/6$

8. Ans. :- $x = \pm\frac{1}{\sqrt{2}}$

9. Ans. :- $\sin^{-1} x - \sin^{-1}\sqrt{x}$

11. Ans. :- ± 3

GROUP–C (6 Marks)

Q.1. :- If the angle C of a triangle ABC be a right angle, Prove that $\tan^{-1}\frac{a}{b+c} + \tan^{-1}\frac{b}{c+a} = \pi/4$ where *a, b, c* are the lenghts of the sides opposite to the angle A, B, C respectively.

Q.2. :- If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{2} = \theta$, Prove that —

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\theta + \frac{y^2}{b^2} = \sin^2\theta$$

Q.3. :- Prove that —

$$\sin^{-1}\frac{x}{\sqrt{1+x^2}} + \cos^{-1}\frac{x+1}{\sqrt{x^2+2x+2}} = \tan^{-1}\left(x^2+x+1\right)$$

Q.4. :- Prove that —

$$\tan^{-1}\left\{\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right\} = \pi/4 + \frac{1}{2}\cos^{-1} x$$

Q.5. :- Write in simplest form —

$$\tan^{-1}\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$$

Q.6. :- Prove — $\cot^{-1}\left[\frac{ab+1}{a-b}\right]+\cot^{-1}\left[\frac{bc+1}{b-c}\right]+\cot^{-1}\left[\frac{ca+1}{c-a}\right]=0$

Q.7. :- If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$

Prove that $x^4+y^4+z^4+4x^2y^2z^2=2\left(x^2y^2+y^2z^2+z^2x^2\right)$

Q.8. :- Prove that—

$$\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{zx}{yr}\right)+\tan^{-1}\left(\frac{xy}{zr}\right)=\pi/2$$

where, $x^2+y^2+z^2=r^2$

Q.9. :- Prove that,

$$\cos^{-1}\left(\frac{\cos x+\cos y}{1+\cos x\cdot\cos y}\right)=2\tan^{-1}\left(\tan\frac{x}{2}\cdot\tan\frac{y}{2}\right)$$

Q.10. :- If two angles of a triangle ABC are $tan^{-1}2$ and $tan^{-1}3$, what is the third angle?

GROUP–C (ANSWERS) (6 Marks)

1. Ans. :- $\left[c^2=a^2+b^2 \quad : \text{ L.H.S. } = \tan^{-1}1=\text{etc.}\right]$

3. Ans. :- [Hint :- Put x $=\tan\theta$ in first and $x+1=\cot\theta$ in the Second term of the L.H.S., then

$$L.H.S.=\sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)+\cos^{-1}\left(\frac{\cot\phi}{\text{cosec}\phi}\right)$$
$$=\sin^{-1}\left(\sin\theta\right)+\cos^{-1}\left(\cos\phi\right)$$
$$=\theta+\phi$$
$$=\tan^{-1}x+\cot^{-1}\left(x+1\right)$$

and then proceed

5. Ans. :- $\frac{x+y}{1-xy}$

CHAPTER–3
Matrices

Key point

1. An ordered rectangular array of numbers (or functions) is called matrix. The numbers are called elements (entries) of the matrix.

Various types of matrices

(a) **Row matrix :** A matrix having only one row.

(b) **Column matrix :** A matrix having only one column.

(c) **Square matrix :** A matrix having equal number of rows and column.

(d) **Diagonal matrix :** A square matrix in which every non-diagonal element is zero is called diagonal matrix.

i.e. A = [aij]n×n be a diagonal matrix, then aij = 0 when $i \neq j$

(e) **Scalar matrix :** $A=[aij]n\times n$ is a scalar matrix if $aij = 0$ when $i \neq j$, $aij = k$ (k is constant) when $i = j$.

(f) **Identity matrix :** $A=[aij]n\times n$ is an identity matrix if $aij = 1$ when $i = j$, $aij = 0$ when $i \neq j$.

(g) **Zero matrix :** A matrix each of whose element is zero is called zero matrix.

(h) **Comparable matrices :** Two matrices A and B are said to be comparable if they are of same order.

2. **Operation on matrices :**

(i) **Addition of Matrices :** if $A=[aij]m\times n$ and $B=[bij]m\times n$ then $A+B=[aij+bij]m\times n$.

(ii) **Negative of a matrices :** Let, $A=[aij]m\times n$. Then the negative of A is the matrix $(-A)=[-aij]m\times n$ obtained by replacing each element of A with its corresponding additive inverse (–A) is called the additive inverse of A.

(iii) If A and B are comparable matrices and K is scaler, then K(A+B) = (KA+KB)

(iv) If K_1, K_2 are scalers and A is any matrix then

(a) $(K_1+K_2)\ A = (K_1A+K_2A)$

(b) $K_1 . (K_2A) = (K_1 . K_2)A$.

(v) **Product of matrices :** Let, $A=[aij]m\times n$ and $B=[bik]n\times p$ be two matrices such that the number of column in A equals the number of rows in B. Then, AB exists and it is an (m×p) matrix, given by

$$AB=[Cij]m\times p \text{ where } Cij=\sum_{j=1}^{n} aij\ bjk$$

3. **Transpose of a matrix :** Matrix obtained by interchanging the rows and column.
4. **Properties of transpose matrix :**

   (i) $\left(A^1\right)^1 = A$ (ii) $(KA)^1 = KA^1$ (iii) $(A+B)^1 = A^1 + B^1$ (iv) $(AB)^1 = B^1A^1$

5. **Symmetric matrix :** A square matrix A is said to be symmetric if $A^1 = A$.
6. **Skew-Symmetric matrix :** A square matrix A is said to be skew-symmetric if $A^1 = -A$.
7. A square matrix can be expressed uniquely as the sum of a symmetric and a skew-symmetric
   matrices.

   Let A be square matrix, then we can write

$$A = \frac{1}{2}\left(A + A^1\right) + \frac{1}{2}\left(A - A^1\right)$$

8. Elementrary operations of a matrix are permissible either on rows or on columns as under :

   (i) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

   (ii) $R_i \leftrightarrow K \cdot R_i$ or $C_i \leftrightarrow KC_i$

   (iii) $R_i \leftrightarrow R_i + R_j$ or $C_i \leftrightarrow KC_j$

9. (i) If A and B are two square matrices such that AB = BA = I then B is inverse of A and is denoted by A^{-1}.

   (ii) Inverse of square matrix, if it exist, is unique.

ILLUSTRATIVE EXAMPLES

Example-1

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, prove that $A^3 - 4A^2 + A = 0$

Solution : We have

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times2+3\times1 & 2\times3+3\times2 \\ 1\times2+2\times1 & 1\times3+2\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

$$A^3 = A^2 \times A$$
$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 7\times2+12\times1 & 7\times3+12\times2 \\ 4\times2+7\times1 & 4\times3+7\times2 \end{bmatrix}$$
$$= \begin{bmatrix} 26 & 45 \\ 15 & 26 \end{bmatrix}$$

$$\therefore A^3 - 4A^2 + A = \begin{bmatrix} 26 & 45 \\ 15 & 26 \end{bmatrix} - 4\begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 26-28+2 & 45-48+3 \\ 15-16+1 & 26-28+2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Example 2 : Find 2×2 matix B such that

$$\begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} B = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$$

Solution : Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then we have $\begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 6a+5c & 6b+5b \\ 5a+6c & 5b+6b \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$$

Compairing, 6a + 5c = 11 .... (i)
6b + 5b = 0 .... (ii)
5a + 6c = 0 .... (iii)
5b + 6b = 11 .... (iv)

Solving (i) & (ii) We get $a = 6, c = -5$

Solving (ii) & (iii) $b = -5 \,\&\, d = 6$

Hence $B = \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix}$

Example.3: Find a matrix X such that

$$2A+B+X=0$$

Where $A=\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix}, B=\begin{bmatrix}3 & -2\\ 1 & 5\end{bmatrix}$

Solution : We have

$$2A+B+X=0$$

$$\Rightarrow X=-2A-B=-2\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix}-\begin{bmatrix}3 & -2\\ 1 & 5\end{bmatrix}$$

$$=\begin{bmatrix}2 & -4\\ -6 & -8\end{bmatrix}+\begin{bmatrix}-3 & 2\\ -1 & -5\end{bmatrix}$$

$$=\begin{bmatrix}2-3 & -4+2\\ -6-1 & -8-5\end{bmatrix}=\begin{bmatrix}-1 & -2\\ -7 & -13\end{bmatrix}$$

Example.4 : It A and B are symmetric matrics, Prove that (AB–BA) is skew symmetric

Solution : $\because$ A and B are symmetric matrics

$\therefore$ $A^1 = A$ and $B^1 = B$

Now

$$(AB-BA)^1=(AB)^1-(BA)^1$$

$$=B^1A^1-A^1B^1$$

$$=BA-AB\left(\text{As } A^1=A \text{ and } B=B^1\right)$$

$$=-(AB-BA)$$

$$Thus,\ (AB-BA)^1=-(AB-BA)$$

$$Hence\,(AB-BA)\,is\ skewsymmetric$$

MATRICES
GROUP-A (1 MARKS)

1. Construct a 2×3 matrix whose elements in the ith row and the jth column are given by $aji=\frac{i+3j}{2}$

2. If $A=\begin{bmatrix}6 & -3\\ -2 & 1\end{bmatrix}$, show that A^{-1} doesnot exist.

3. Using elementary row transformation, find the inverse of the following

$$\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$$

4. Simplify : $Cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}+\sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

5. Find X and Y, if

$$X+Y=\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} and\ X-Y=\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

6. If $A=\begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$, show that $A-A^T$ is skew symmetric matrix, where A^T is the transpose of matrix A.

7. If A and B are symmetric matrices, prove that (AB – BA) is skew symmetric.

8. If $A=\begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 4 & -6 \end{bmatrix}$, verify that $(2A)^1=2A^1$

9. Show that $AB\neq BA$

$$A=\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} and\ B=\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

10. If $A=\begin{bmatrix} \alpha & \beta \\ \gamma & \alpha \end{bmatrix}$ is such that $A^2=I$

$(a) 1+\alpha^2+\beta\gamma=0$ $(b) 1-\alpha^2+\beta\gamma=0$

$(c) 1-\alpha^2-\beta\gamma=0$ $(d) 1+\alpha^2-\beta\gamma=0$

11. If $A=[a_{ij}]$, where $a_{ij}=\begin{Bmatrix} i+j, & if & i\geq j \\ i-j, & if & i\geq j \end{Bmatrix}$, contruct a 3×2 matrix A.

GROUP–A (1 MARKS) (ANSWER)

1. Ans. : $\begin{bmatrix} 2 & 7/2 & 5 \\ 5/2 & 4 & 11/2 \end{bmatrix}$

2. Ans. : $\frac{1}{17}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$

3. Ans. : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. Ans. : $X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

10. Ans. : (c)

11. Ans. : $\begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$

GROUP–B (4 marks)

1. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, then find K such that $A^2 - 8A + KI = 0$.

2. If $A = \begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix}$, find $f(A)$, where $f(x) = x^2 - 5x + 7$.

3. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$ for all positive integers n.

4. If $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}$, Verify that $(AB)^1 = B^1A^1$.

5. Express the following matrices as the sum of a summetric and a skew-symmentric matrices.

$$\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$$

6. Find x, y, z, w if :

$$\begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix} = 3\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

7. Let $A = \begin{bmatrix} 0 & -\tan \alpha/2 \\ \tan \alpha/2 & 0 \end{bmatrix}$ and I is the identity matrix of order 2. Show

that $(I+A)=(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

8. If $X=\begin{bmatrix}\cos A & \sin A\\ -\sin A & \cos A\end{bmatrix}$, prove that $X^{\eta}=\begin{bmatrix}\cos nA & \sin nA\\ -\sin nA & \cos nA\end{bmatrix}, \eta\in N$

9. If $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $E=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, prove that $(aI+bE)^3=a^3I+3a^2bE$.

10. Let $A=\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}5 & 2\\ 7 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 5\\ 3 & 8\end{bmatrix}$ find a matrix D such that CD – AB = 0.

GROUP–B (4 MARKS) (ANSWER)

1. Ans. :- 7

2. Ans. :- $\begin{bmatrix}-15 & -20\\ 20 & 15\end{bmatrix}$

5. Ans. :- $\begin{bmatrix}3 & 3\\ 3 & -1\end{bmatrix}+\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$

6. Ans. :- $x = 2, y = 4, w = 3, z = 1$.

10. Ans. :- $D=\begin{bmatrix}-191 & -110\\ 77 & 44\end{bmatrix}$

GROUP–C (6 MARKS)

1. If $A^1=\begin{bmatrix}3 & 4\\ -1 & 2\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}$, then show that :—

   (i) $(A+B)^1=A^1+B^1$ (ii) $(A-B)^1=A^1-B^1$

2. A trust fund has Rs. 30,000 that different types of bonds. The first bond pays 5% interest per year and the second bond pays 7% interest per year. Using matrix multiplication, find how to divide Rs. 30000 among two types of bonds, if the trust fund must obtain an annual total interest of (i) Rs. 1,800 (ii) Rs. 2,000.
3. Express the following matrices as the sum of symmetric and skew-symmetric matrices.

$$\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$$

4. If $A=\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$; $B=\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ then show that AB is a zero matrix, provided $(\theta-\phi)$ is an odd multiple of $\pi/2$.

5. Find a 2 × 2 matrix N such that : $N\begin{bmatrix} -7 & 4 \\ 5 & -8 \end{bmatrix}=\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$

6. In two families A and B, A consists of 3 men, 4 women and 5 children and B consists of 2 men, 3 women and 3 children. Expected daily expenses on lunch is Man–Rs. 45, Women–Rs. 40, Child–Rs. 30 and on Dinner is Man–Rs. 50, Women–Rs. 30 and child–Rs. 30. Represent the given information by matrices. Using matrix multiplication, compute the total daily expenditures on lunch and dinner for each of the two families.

7. If A and B are square matrices of the same order, such that AB=BA, then prove by induction that $AB^n = B^nA$. Further prove that $(AB)^n = A^nB^n$ for all $n \in N$.

8. If $A=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, prove that $A^n=\begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$ for all positive integer *n*.

9. Show that all positive odd integral powers of a skew-symmentric matrix are skew symmetric all positive even integral powers of a skew-symmetric matrix are symmetric.

10. If $A_\alpha=\begin{bmatrix} \cos a & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ show that $(A_\alpha)^n = A_{n\alpha}$, $h = N$.

GROUP–C (6 MARKS) ANSWERS

2. Ans. :- (i) Rs. 15000, Rs. 15000, (ii) Rs. 5000, Rs. 25000

3. Ans. :- $\begin{bmatrix} 2 & -3/2 & -3/2 \\ -3/2 & 3 & 1 \\ -3/2 & 1 & -3 \end{bmatrix}$

6. Ans. :- Family A on lunch and dinner are respectively Rs. 445 and Rs.370. Family B on lunch and dinner are respectively Rs. 300 and Rs. 250.

CHAPTER–4

DETERMINANTS

Key point

1. **Determinant :** To each square matrix $A=[aij]$ of order *n*, there is associated a number (Real or Complex), which is called determinants of square matrix A.
2. **Determinants :** A matrix of order one.

   Let, $A=[a]$ be a matrix of order 1 × 1.

   Then det $A=|A|=|a|=a$.
3. **Determinant of a matrix of order two :**

   Let, $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2 × 2.

   Then det $A=|A|=\Delta=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

   $=a_{11}\cdot a_{22}-a_{12}\cdot a_{21}$
4. **Determinant of a matrix of order three :**

   Let, $A=[aij]_{3\times3}$ be a matrix of order 3 × 3

   Then, $|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

   There are six ways of expanding a determinant of order 3 × 3.
5. **Properties of determinants :**

   a. The values of a determinants remain uncganged if its rows and columns are interchanged.

   b. If two rows or comlumns of a determinant are interchanged then determinant retains its absolute value but sign is changed.

   c. If any two rows or columns of a determinants are identical then its value is zero.

   d. If each element of a row or column of a determinant is multiplied by a constant K thenthe value of the new determinant is K times the value of the original determinant.

   e. If each element of a row (or column) of a determinant is expressed as a sum of the two or more terms then the determinant can be expressed as the sum of two or more determinants.

   f. If to any row or column of a determinant, a multiple of another row or column is added, the value of the determinant remains the same.

6. **Minor of a_{ij} in |A| :**
   The minor of an element a_{ij} in |A| is defined as the values of the determinant obtained by deleting the i^{th} row and j^{th} column of |A| and it is denoted by M_{ij}.
7. **Co-factor of a_{ij} in |A| :** The co-factor A_{ij} of an element a_{ij} is defined as $a_{ij} = (-1)^{i+j} M_{ij}$.
8. (i) A square matrix A is said to be singular matrix, if |A| = 0.
   (ii) A square matrix A is said to be non-singular matrix if $|A| \neq 0$.
9. inverse of a non singular matrix A can be obtained by $A^{-1} = \frac{1}{|A|}$ (Adj A)
10. Area of triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

11. Matrix method to solve system of linear equations :-

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then these equation can be written as
AX = B, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

12. Unique solution of equation AX = B is given by $X = A^{-1} \cdot B$, where $|A| \neq 0$.
13. A system of equation is consistent or inconsistent according as its solution exist or not.
14. For a square matrix A in a matrix equation AX = B.
    (i) $|A| \neq 0$, there exists a unique solution.
    (ii) |A| = 0, and (adj A) B $\neq$ 0.
    In this case, the given system has no solution and hence it is in consistent.
    (iii) When |A| = 0 and (adj A) B = 0.
    In this case, the given system has intinity many solution.

ILLUSTRATIVE EXAMPLES

Example-1 : Without expanding, prove that

$$\begin{bmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{bmatrix} = 0$$

Solution : Let the given determinant be D. Then

$$\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} \quad (R_1 \to R_1 + R_2)$$

Taking $(x + y + z)$ common from R_1

$$\Delta = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (x+y+z)\times 0 \qquad (R_1 \,\&\, R_3 \text{ are identical})$$

$$= 0.$$

Example-2 : Prove that $\begin{bmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{bmatrix} = bc + ca + ab + abc$

Solution : The given determinant

$$= \begin{bmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{bmatrix}$$

Tacing a, b, c common from R_1, R_2 and R_3 respectively.

$$= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1+R_2+R_3$

$$= abc \begin{vmatrix} \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $c_1 \rightarrow c_1 - c_2$ and $c_2 \rightarrow c_2 - c_3$

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & \frac{1}{c}+1 \end{vmatrix}$$

Expanding by first row

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right) 1 \times \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}$$

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)$$

$$= abc\left(\frac{bc+ac+ab+abc}{abc}\right)$$

$= bc + ac + ab + abc$

Example :- Using matrices, solve the following system of linear equations :

$$3x + 4y + 2z = 8$$
$$2y - 3z = 3$$
$$x - 2y + 6z = -2$$

Solution : The given equations are

$$3x + 4y + 2z = 8 \quad \text{---------------- (i)}$$
$$2y - 3z = 3 \quad \text{---------------- (ii)}$$
$$x - 2y + 6z = -2 \quad \text{---------------- (iii)}$$

Let A= $\begin{bmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}$

So, the given system in matrix term is AX = B

Now |A| = $\begin{vmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{vmatrix}$

$= \begin{vmatrix} 0 & 10 & -16 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{vmatrix} [R_1 \to R_1 - 3R_2]$

= 1 × (–30 + 32) = 2 ¹ 0

So, A is invertible.

Now, The co-factor of the element of |A| are

$A_{11} = 6$ $\quad A_{12} = -3$ $\quad A_{13} = -2$

$A_{21} = -28$ $\quad A_{22} = 16$ $\quad A_{23} = 10$

$A_{31} = -16$ $\quad A_{23} = 9$ $\quad A_{33} = 6$

$$\therefore \text{(adj A)} = \begin{bmatrix} 6 & -3 & -2 \\ -28 & 16 & 10 \\ -16 & 9 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -28 & -16 \\ -3 & 16 & 9 \\ -2 & 10 & 6 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\,(adj\ A)$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -28 & -16 \\ -3 & 16 & 9 \\ -2 & 10 & 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -14 & -8 \\ -3/2 & 8 & 9/2 \\ -1 & 5 & 3 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -14 & -8 \\ -3/2 & 8 & 9/2 \\ -1 & 5 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$\therefore$ $x = -2, y = 3, z = 1.$

DETERMINANTS
GROUP–A (1-marks)

Q.1. If A is an invertible matrix of order 3 × 3. Then |adj A| is equal to

(i) |A| (ii) $|A|^2$ (iii) $|A|^3$ (iv) 3|A|

Q.2. For a square matrix A of order *n*, |KA| is equal to ________________.

Q.3. If *a, b, c* are in A.P., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} \text{ is}$$

(i) 0 (ii) 1 (iii) x (iv) $2x$

Q.4. Evaluate – $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

Q.5. A is non-singular matrix of order 3 and |A| = –4. Find |*adj* A|.

Q.6. Given determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ find the value of $a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{33}$

Q.7. Find the value of |A|, where A = $\begin{vmatrix} \sin 10^0 & -\cos 10^0 \\ \sin 80^0 & -\cos 80^0 \end{vmatrix}$

Q.8. If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Q.9. Using the property of determinants and without expanding, prove that —

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Q.10. Prove that —

$$\begin{vmatrix} 1 & x & x^2 - yz \\ 1 & y & y^2 - zx \\ 1 & z & z^2 - xy \end{vmatrix} = 0$$

GROUP–A (ANSWERS) (1 MARKS)

1. Ans. :- (ii)
2. Ans. :- K^n |A|
3. Ans. :- (i)
4. Ans. :- $a^2 + b^2 + c^2 + d^2$
5. Ans. :- 16.
6. Ans. :- 0
7. Ans. :- 1
8. Ans. :- $B^{-1}A^{-1}$

GROUP–B (4-marks)

Q.1. Prove that – $\begin{vmatrix} a+l & m & n \\ l & a+m & n \\ l & m & a+n \end{vmatrix} = a^2(a+l+m+n)$

Q.2. Prove that – $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

Q.3. Show that the matrix $A=\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$, satisfies the equation $A^2-5A+7I=0$,

Hence find A^{-1}

Q.4. Show that – $\begin{vmatrix}-\alpha^2 & \gamma\beta & \gamma\alpha\\ \alpha\beta & -\beta^2 & \beta\gamma\\ \gamma\alpha & \beta\gamma & -\gamma^2\end{vmatrix}=4\alpha^2\beta^2\gamma^2$

Q.5. Using matrices solve the following system of .... equations :–

$$x+3y+4z=8$$
$$2x+y+2z=5$$
$$5x+y+z=7$$

Q.6. Using properties of derminants, prove that –

$$\begin{vmatrix}m_{c_1} & m_{c_2} & m_{c_3}\\ n_{c_1} & n_{c_2} & n_{c_3}\\ p_{c_1} & p_{c_3} & p_{c_3}\end{vmatrix}=\frac{mpn(m-n)(n-p)(p-m)}{12}$$

Q.7. If x,y,z, are non-zero real numbers, then find A^{-1} where,

$$A=\begin{vmatrix}x & o & o\\ o & y & o\\ o & o & z\end{vmatrix}$$

Q.8. Prove that –

$$\begin{vmatrix}a+bx & c+dx & p+qx\\ ax+b & cx+d & px+q\\ u & v & w\end{vmatrix}=\left(1-x^2\right)\begin{vmatrix}a & c & p\\ b & d & q\\ u & v & w\end{vmatrix}$$

Q.9. Prove $\begin{vmatrix}x & y & z\\ x^2 & y^2 & z^2\\ x^3 & y^3 & z^3\end{vmatrix}=xyz(x-y)(y-z)(z-x)$

Q.10. Compute the adjoint of matrix $A=\begin{vmatrix}1 & 2\\ 3 & -5\end{vmatrix}$ and verify that $A(adj\ A)=|A|I$.

GROUP–B (ANSWERS) 4 MARKS)

5. Ans. :- $x = 1, y = 1, z = 1$

7. Ans. :- $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

10. Ans. :- $adj\ A = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$

GROUP – (6 marks)

Q.1. If a,b, & c, all positive are p^{th}, q^{th} and r^{th} teams of a G.P.

Prove that $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$

Q.2. Find the invertible of the matrix, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

Q.3. If $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that $(adjA)^{-1} = (adjA^{-1})$

Q.4. Using properties of derminants, prove that –

$$\begin{bmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2-c^2) \end{bmatrix} = abc(a^2+b^2+c^2)^3$$

Q.5. If a,b,c are positive unequal, show that the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

Q.6. Show that – $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

Q.7. The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

Q.8. Show that – $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$$= abc+bc+ca+ab$$

Q.9. Show that $\triangle = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$

Q10. Show that $\begin{vmatrix} a & b & a\alpha+c \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$, if either a,b,c are in G.P. or α is a root of the equation $ax^2+2bx+c=0$

GROUP – C (ANSWERS) (6 – marks)

2. Ans :– $A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

7. Ans :– $x = 1, y = 2, z = 3$

CONTINUITY AND DIFFERENTIABILITY

SYNOPSIS

1. Let f (n) be a real valued function on the subset of real numbers and let 'a' lr any point in the domain of 'f'. Than f is said to continuous at $x = a$ if

   $\lim f(x) = f(a)$
   $x \to a$

   i.e. the value of the function at $x = a$ is same as limit of the function f as $x \to a$
   We can also say that f is continuous at $x = a$ if

   LHS = RHS = functional value
   at a   at a

   $$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

2. A function is discontinuous at $x = a$ if
   (i) $f(a)$ doesnot exists (ii) lim f (x) does not existt (iii) Both exist but are not equal
3. A function f is said to be continuous if it is continuous at every point in the domain of 'f'.
4. Let f and give continuous functions at $x = a$ than
   (a) α f is continuous at $x = a$
   (b) f + g is continuous at $x = a$
   (c) f - g is continuous at $x = a$
   (d) f .g is continuous at $x = a$
   (e) $\frac{f}{g}$ is continuous at $n = a$ provided g(c)$\neq 0$
5. Differentiability at a point :

   Let f (x) be a real valued function defined on anopen interval (a,b) and let $c \in (a,b)$. Then f (x) is said to be diffentiable or derivable at $x = c$ if and only if

   $$\lim \frac{f(x) - f(c)}{x - c}$$

   exists finitely which is doneted by $f^1(c)$.

   We can also say that $f(c)$ is differentable at $x = c$

LHD at $x=c$ RHD at $x=c$

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}=\lim_{x-c^-}\frac{f(x)-f(c)}{x-c}$$

Th : If a function at $x=c$ is diffentiable, it is necessary continous at that point. But the convers is not necessarily true.

Following are derivatives of some standard functions :

1. $\frac{d}{dx}\left(x^n\right)=n\cdot x^{n-1}$
2. $\frac{d}{dx}\left(e^x\right)=e^x$
3. $\frac{d}{dx}\left(a^x\right)=a^x\cdot\log_e a$
4. $\frac{d}{dx}\left(\log_e^x\right)=\frac{1}{x}$
5. $\frac{d}{dx}\left(\log_a^x\right)=\frac{1}{x\log_e^a}$
6. $\frac{d}{dx}\left(Const\right)=0$
7. $\frac{d}{dx}\left(K\cdot f(x)\right)=K\cdot\frac{d}{dn}f(n)$
8. $\frac{d}{dx}\left(Sinx\right)=Cosx$
9. $\frac{d}{dx}\left(Cosx\right)=-Sinx$
10. $\frac{d}{dx}\left(\tan x\right)=Sec^2x$
11. $\frac{d}{dx}\left(\cot x\right)=-\text{ Cosec}x^2$
12. $\frac{d}{dx}\left(\text{Sec}x\right)=\text{Sec}x\tan x$
13. $\frac{d}{dx}\left(\text{Cosec}x\right)=-\text{Cosec}x\text{ Cot}x$
14. $\frac{d}{dx}\left(\text{Sin}^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$

15. $\frac{d}{dx}\left(\text{Cos}^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}$

16. $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$

17. $\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$

18. $\frac{d}{dx}\left(\text{Sec}^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}$

19. $\frac{d}{dx}\left(\text{Cosec}^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}$

20. $\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)$

21. $\frac{d}{dx}[f(x)\cdot g(x)]=f(x)\cdot g^{1}\left(x\right)+g(x)\cdot f^{1}(x)$ (Product rule)

22. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot f^{1}\left(x\right)-f(x)\cdot g^{1}(x)}{\left\{g\left(x\right)\right\}^2}$ (Quotient rule)

23. Chain rule :

Let $y:f(u)$ and $u=g(x)$ then $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

24. Parametric differentiation :

If $y=f(t)$, $x=g\left(t\right)$, t is a parametric then

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}\Rightarrow\frac{dy}{dt}\times\frac{dt}{dx}$$

25. Second order derivatives :

If $y=f\left(x\right)$ then $\frac{d}{dx}\left[\frac{dy}{dx}\right]$ is called second order derivative of with respect to *x* and is denoted by $\frac{d^2y}{dx^2}$ or $f^{1}\left(x\right)$.

26. If $x=f\left(t\right)$, $y=g\left(t\right)$, then

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{g^{1}\left(t\right)}{f^{1}\left(t\right)}\right]=\frac{d}{dt}\left[\frac{g^{1}\left(t\right)}{f^{1}\left(t\right)}\right]\cdot\frac{dt}{dx}$$

27. If $y = f(x),\ g(x),\ u(x)$ and $v(x)$ are function of x and D is determinant given by

$$\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}$$

Then,

$$\frac{d}{dx}\{\Delta(x)\} = \begin{vmatrix} f^1(x) & g^1(x) \\ u(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ u^1(x) & v^1(x) \end{vmatrix}$$

OR

$$= \begin{vmatrix} f^1(x) & g^1(x) \\ u^1(x) & v^1(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}$$

Similar results hold for the differentiation of determinants of higher order.

28. **Rolle's theorem :**
If a function is defined in [a, b] such that
i) *f* is continuous inversed interval [a, b]
ii) *f* is derivative in open interval (a, b)
iii) *f(a)* = *f(b)* then there exists at least one one c ∈ (a, b) such that $f^1(c) = 0$.

29. **Lagrange's Mean Value Theorem :** If a function is defined in (a, b) such that
i) *f* is continuous in (a, b)
ii) *f* is differentiable in (a, b) then there exists c Î (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

ILLUSTRATIVE EXAMPLES :

1. Show that the function *f(x)* is given by

$$f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1} & When\ x \neq 0 \\ & When\ x = 0 \end{cases}$$ is discontinuous $x = 0$

Solution : We have

LHL at $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = f(-h)$$

$$= \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = \lim_{h \to 0} \frac{\frac{1}{e^{1/h}} - 1}{\frac{1}{e^{1/h}} + 1} = \frac{0 - 1}{0 + 1} = -1$$

and RHL at $x = 0$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+n) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1} = \lim_{h\to 0} \frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{1/h}}} = \frac{1-0}{1+0} = 1$$

so LHL ≠ RHL

∴ *f(x)* is discontinuous at $x = 0$

2. Discuss the differentiability of $f(x) = |x–1| + |x-2|$

Solution :

We have $f(x) = |x-1| + |x-2|$

$$f(x) = \begin{cases} -(x-1)-(x-2) & \text{for } x<1 \\ x-1-(x-2) & \text{for } 1\le x<2 \\ x-1+(x-2) & \text{for } x\ge 2 \end{cases}$$

$$\therefore \quad f(x) = \begin{cases} -2x+3 & , \quad x<1 \\ 1 & , \quad 1\le x<2 \\ 2x-3 & , \quad x\ge 2 \end{cases}$$

$$(\text{LHD at } x=1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{(-2x+3)-1}{x-1}$$

$$= \lim_{x\to 1} \frac{-2(x-1)}{x-1} = -2$$

$$(\text{RHD at } x=1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{n\to 1^+} \frac{1-1}{x-1} = 0$$

LHD ≠ RHD

So *f(x)* is not differentiable at $x = 1$

$$(\text{LHD at } x=2) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{1-(2\times 2-3)}{x-1} = \frac{1-1}{x-2} = 0$$

$$(\text{RHD at } x=2) = \lim_{x\to 2^+} \frac{f(x)-f(1)}{x-2} = \lim_{x\to 2} \frac{2x-4}{x-1} = \frac{2(x-4)}{x-2}2$$

LHD ¹ RHD

So $f(x)$ is not differentiable at $x = 2$

Remark : It should be noted that the function $f(x)$ is given by

$f(x) = |x-a_1| + |x-a_2| + |x-a_3| + \text{----------} + |x-a_n|$ is not differentiable at $x = a_1, a_2, a_3, a_4, \text{---------}\ a_n$

3. If $y = \left(x+\sqrt{x^2+a^2}\right)^n$ then prove that $\dfrac{dy}{dx} = \dfrac{ny}{\sqrt{x^2+a^2}}$.

Solution : We have

$$y = \left(x+\sqrt{x^2+a^2}\right)^n$$

$$\frac{dy}{dx} = n\left[x+\sqrt{x^2+a^2}\right]^{n-1} \cdot \frac{d}{dx}\left[x+\sqrt{x^2+a^2}\right]$$

$$\frac{dy}{dx} = n\left[x+\sqrt{x^2+a^2}\right]^{n-1} \cdot \left[1+\frac{1}{2\sqrt{x^2+a^2}} \cdot \frac{d}{dx}\left(x^2+a^2\right)\right]$$

$$= n\left[x+\sqrt{x^2+a^2}\right]^{n-1} \cdot \left[1+\frac{1}{\cancel{2}\sqrt{x^2+a^2}} \cdot \cancel{2}x\right]$$

$$= n\left[x+\sqrt{x^2+a^2}\right]^{n-1} \cdot \left[1+\frac{n}{x^2+a^2}\right]$$

$$= n\left[x+\sqrt{x^2+a^2}\right]^{n-1} \cdot \left[\frac{\sqrt{x^2+a^2}+x}{x^2+a^2}\right]$$

$$= \frac{n\left[x+\sqrt{x^2+a^2}\right]^{n-\cancel{1}+\cancel{1}}}{\sqrt{x^2+a^2}} = \frac{ny}{\sqrt{n^2+a^2}}$$

4. Differentiate $\tan^{-1}\sqrt{\dfrac{1+\sin x}{1-\sin x}},\ \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$.

Solution : Let $y = \tan^{-1}\sqrt{\frac{1+\sin x}{1-\sin x}}$

$$= \tan^{-1}\sqrt{\frac{1+Cos\left(\frac{\pi}{2}-x\right)}{1-Cos\left(\frac{\pi}{2}-x\right)}}$$

$$= \tan^{-1}\sqrt{\frac{\not{2}Cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\not{2}Sin^2\left(\frac{\pi}{2}-\frac{x}{2}\right)}}$$

$$= \tan^{-1}\sqrt{Cot^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}$$

$$= \tan^{-1}\left(Cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \quad \text{as} \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$$

$$0 < \frac{x}{2}+\frac{\pi}{4} < \frac{\pi}{2}$$

$$y = \frac{\pi}{4}+\frac{x}{2}$$

$$\therefore \frac{dy}{dx} = 0 + \frac{1}{2} = \frac{1}{2}$$

5. If $x^m y^n = (x+y)^{m+n}$ Prove that $\frac{dy}{dx} = \frac{y}{x}$

Solution : $x^m . y^n = (x+y)^{m+n}$

Take log on both sides

$\log (x^m \times y^n) = \log\left((x+y)^{m+n}\right)$

$\log x^m + \log y^n = (m+n) \log (x+y)$

diff wr+x

$$\frac{m}{x}+\frac{n}{y}.\frac{dy}{dx}=(m+n).\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx}=\frac{m+n}{x+y}-\frac{m}{x}$$

$$\left(\frac{n(x+y)-y(m+n)}{y(x+y)}\right)\frac{dy}{dx}=\frac{(m+n)x-m(x+y)}{(x+y)x}$$

$$\frac{nx-my}{y(m-y)}\frac{dy}{dx}=\frac{nx-my}{(m-y)}$$

$$\frac{dy}{dx}=\frac{y}{x}$$

6. Verity the lagrangis mean value therum for the function :

f(x) = 2sin x + sin 2x on $[0,\pi]$

Solution :

Since sin 2x, sin x are every where continous and differentiable

$\therefore$ these $f(x) = 2\sin x + \sin 2x$ is also continous in $[0,\pi]$ and differentable in $[0,\pi]$

f(x) = 2sin x + sin 2x

$f^1(x) = 2\cos x + 2\cos 2x$

f(o) = 0 and $f(\pi) = 2\sin\pi + \sin 2\pi = 0$

$$\therefore\ f^1(x)=\frac{f(\pi)-f(0)}{\pi-o}$$

$$2\cos x = 2\cos 2x = 0 \Rightarrow \cos x+\cos 2x=0$$

$$\cos 2x = -\cos x$$

$$\cos 2x=\cos(\pi-x)$$

$$2x=\frac{\pi}{3}\in(0,\pi)\text{ such that } f(c)=\frac{f(\pi)-f(o)}{\pi-0}$$

Problems for Practice
Section A (one mark)

1. Discuss the continuity of $f(x)=\begin{cases}\frac{1-\cos 2x}{x^2} & x\neq 0 \text{ at x = 0}\\ =5 & , \quad x=0\end{cases}$

2. Find the derivative of x^x

3. Find the derivative of $\log_{10}(\sin x)$

4. If $x = at^2$, $y = 2at$ find $\dfrac{dy}{dx}$

5. Find $\dfrac{dy}{dx}$ if $y = \log(\cos x^2)$

6. Find $\dfrac{dy}{dx}$ if $y = 5\log\sin x$

7. Find the derivation of $\cos^{-1}(\sin x)\ wrt\ \ x$

8. If $y = x^2 + xy + y^2 = 100\ find\ \dfrac{dy}{dx}$

10. If a function is differecetable at every point it is (...) that .... at that point.

Group B (4 marks)

1. Determine the value of a, b and c for which the function

$$f(x) = \begin{cases} = \dfrac{\sin(a+1)x + \sin x}{x}, & x < 0 \\ = \quad C, & x = 0 \\ = \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{b\sqrt{x^3}}, & x > 0 \end{cases}$$ may be continous at $x = 0$

2. Let $f(x) = \dfrac{1-\cos 4x}{x^2}, \quad x < 0$

$$= a \qquad x = 0$$

$$\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4}, \quad x > 0$$

Determine the value of ‘a’ if possible so that the function is continuous at $x = 0$

3. Find $\dfrac{dy}{dx}$ if $y = \tan^{-1}\left(\dfrac{\text{Sin}x}{1-\text{Cos}x}\right)$

4. Find $\dfrac{dy}{dx}$ if $y = e^{3x}\ \log\left(\text{Sin}^2 x\right)$

5. Find $\dfrac{dy}{dx}$ if $x\tan x + (Sinx)^{Cosx}$

6. Find $\frac{dy}{dx}$ if $y=\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

7. If $x = a\,(\text{Cos}\theta + \theta\ \text{Sin}\theta)$, $y = a\,(\sin\theta - \theta\ \cos\theta)$ then find $\frac{dy}{dx}$

8. If $x^y = e^{x-y}$ P.T. $\frac{dy}{dx} = \log x \Big/ (1+\log x)^2$

9. If $x = \left(Cos\theta + \log\tan \theta/2\right)$, $y = \text{Sin}\theta$ find $\frac{d^2y}{dx^2}$ of $\theta = \pi/4$

10. If $(a+bx)^{y/x} = x$ then prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$

11. If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ S.T. $\frac{d^2y}{dx^2} = -b^4 \Big/ a^2y^3$

12. If $y=\left(\tan^{-1}x\right)^2$ prove that $\left(1+x^2\right)^2\frac{d^2y}{dx^2}+2x\left(1+x^2\right)\frac{dy}{dx}=2$

13. Prove that $\frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\text{Sin}^{-1}\frac{x}{a}\right)=\sqrt{a^2-x^2}$

14. If $y = e^{ax}\text{Sin}bx$ then prove that $\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+\left(a^2+b^2\right)y=0$

15. If $y=\sqrt{x}+\frac{1}{\sqrt{x}}$ show that $2x\frac{dy}{dx}+y=2\sqrt{x}$

16. If $y = Ae^{mx}+Be^{nx}$ prove that $\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny=0$

17. If $y=\sqrt{Sinx+\sqrt{Sinx+.....\infty}}$ prove that $\frac{dy}{dx}=\frac{\text{Cos}x}{2y-1}$

18. If $\sin x = y\sin(x+b)$ show that $\frac{dy}{dx}=\frac{\sin b}{\sin^2(x+b)}$

19. If $y=\frac{3at}{1+t}$, $x=\frac{2at^2}{1+t}$, find $\frac{d^2y}{dx^2}$

20. a) Verify the Rolle's theorem for

    i) $f(x)=x^2-5x^2-3x$ in [1, 3]

    ii) $f(x)=x^2$ in (–1, 2)

    b) Verify languages Mean value theorem for

i) $f(x)=x^2-5x^2-3x$ in $[1, 3]$

ii) $f(x)=x(x-1)(x-2)$ on $\left[0, \frac{1}{2}\right]$

GROUP–C (6 MARKS)

1. If $x=\log t$ and $y=\frac{1}{t}$ then prove that $\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$
2. If $y=\left(\sin^{-1}x\right)^2$, prove that $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$
3. If $y=\cot x+\text{cosec}x$ prove that $(1-\cos x)^2\frac{d^2y}{dx^2}=\sin x$
4. If $y=3e^{2x}+2e^{3x}$ then prove that $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$
5. If $3\cos(\log x)+4\sin(\log x)$ prove that $x^2y_2+xy_1+y=0$
6. If $y=\sqrt{\frac{1-\sin 2x}{1+\sin 2x}}$ show that $\frac{dy}{dx}+\sec^2\left(\frac{\pi}{4}-x\right)=0$
7. If $y=(\log x)^2$ then prove that $x^2y''+xy'=2$
8. If $y=e^x\tan^{-1}x$ then prove that $\left(1+x^2\right)\frac{d^2y}{dx^2}-2\left(1-x+x^2\right)\frac{dy}{dx}+(1-x)^2y=0$
9. If $f:[-5,5]\rightarrow R$ is differentiable function and $f'(x)$ does not vanish everywhere prove that $f(-5)\neq f(5)$.
10. Does there exists a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer. $\left(\text{hint}: f(x):|x-1|+|x-2|\right)$

ANSWERS
GROUP–A (1 MARKS)

1. *f(x)* is discontinuous at $x = 0$.

2. $\frac{1}{\sin x\log_e^a}\cdot\cos x$

3. $x^x(1+\log x)$

4. $\frac{1}{t}$

5. $\dfrac{2x\left(-\sin x^2\right)}{\cos x^2}$

6. $5^{\log\sin x}\cdot\cot x$

7. **–1**

8. $2x + x\,{dy}/{dx} + y\cdot 1 + 2y\,{dy}/{dx} = 0 \Rightarrow \dfrac{dy}{dx}(x+2y) = -(2x+y)$

$$ {dy}/{dx} = -{(2x+y)}/{(x+2y)} $$

9. $\dfrac{dy}{dx} = -\dfrac{y}{x}$

10. Continuous

GROUP–A (4 MARKS)

1. $a = \dfrac{3}{2},\ b = \text{any value }(\neq 0)$ and $c = \dfrac{1}{2}$

2. $a = 8$

3. $-\dfrac{1}{2}$

4. $e^{3x}\left[3\log\sin x + 2\cot 2x\right]$

5. $x^{\tan x}\left(\dfrac{\tan x}{x} + \sec^2\cdot\log x\right) + (\sin x)^{\cos x}\left(\dfrac{\cos^2 x}{\sin x} - \sin x\cdot\log\sin x\right)$

6. $\dfrac{1}{2\sqrt{1-x^2}}$

7. $\tan\theta$

8. —

APPLICATIONS OF DERIVATIVES

SYNOPSIS :

1. If a quantity y varies with another quantity 'x' satisfying some rule $y = f(x)$ then $\frac{dy}{dx}$ represents the rate of change of y w.r.t. x and $\frac{dy}{dx}$ at $x = x_o$ represent rate of change of y w.r.t x at $x = x_0$.
2. A functions f is said to

   a) Increasing on an interval (a,b) if $x_1 < x_2$ in (a,b)

   $\Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in$ (a,b)

   Alternatively, if $f'(x) \geq 0$ for each x in (a,b)

   b) Decreasing on (a,b) if $x_1 < x_2$ in (a,b) $\Rightarrow f(x)$ ³ $f(x_2)$ for all $x_1\ x_2 \in (a, b)$.

   Alternately, if $f'(x) \geq 0$ for each x in (a, b).
3. The equation of the tangent at $(x_o,\ y_o)$ to the curve $y = f(x)$ is given by $y - y_o = \frac{dy}{dx}\Big/(x_o, y_o)(x - x_o)$.
4. If $\frac{dy}{dx}$ doesn't exits at the point (xo, yo) then the tangent at this point is parallel to the y-axis and its equation is $x = xo$.
5. If tangent to a curve $y = f(x)$ at $x = xo$ is parallel to x-axis then $\frac{dy}{dx}$ at $x=xo=0$.
6. Equation of the normal to the $y = f(x)$ at a point (x_0, y_0) is given by $y - y_o = \frac{-1}{dx\big/dy(x_o, y_o)}(x - x_o)$
7. It $\frac{dy}{dx}$ at the point (xo, yo) is zero then the equatiuon of the normal is $x = xo$.
8. Let $y = f(x)$, Δx be a small increment in x and Δy be the increment is y corresponding to the increment is x is $\Delta y = f(x + \Delta x) - f(x)$ then dy given by $dy = f^1(x)dx$ or $dy = \left(\frac{dy}{dx}\right).\Delta x.$ is a good approximation of Δy when $dx = \Delta x$ is reclatively small and we denote by $dy \cong \Delta y$.

9. A point C in the domain of the function f at which either $f'(c)=0$ or f is not differentiable is called a critical point of f.

10. **First derivative test :** Let f be a function defined on an open interval I. Let f be a continuous of a critical point 'c' in I. Then

    i) If $f'(x)$ changes sign from positive to negative as x increases through 'c' if $f'(x)>0$ at every point sufficiently close to and to the left of c. and $f'(n)>0$ at every point sufficiently close to and to the right of c. then c is a point of local maxima.

    ii) If $f'(x)$ changes sign from negative to positive as x increase through c. if $f'(x)<0$ at every point suficiently close to and to the left of c and $f'(x)>0$ at every point sufficiently close to and to the right of c. then c is called point of local minima.

    iii) If $f'(x)$ does not change sign as x increases through c then c is neither a point of local maxima nor a point of local minima. Infact such a point is called point of inflection.

Second derivative test :

Let f be a function defined on an internal I and $c \in$ I. Let f be twice differentiable at c then

i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

ii) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) > 0$

in the above *f(c)* is called local maximum/minimum

iii) The test fails if $f'(c)=0$ and $f''(c)=0$ in this case we go back to the first derivative test and find whether c is a point of maxima, minima or a point of influection.

Absolute Maxima/mimimun in closed interval

i) Find all critical point of f in the interval find points x where either $f'(x)=0$ or f is not differentiable.

ii) Take the end point of the interval.

iii) At all these points calculate the values of f

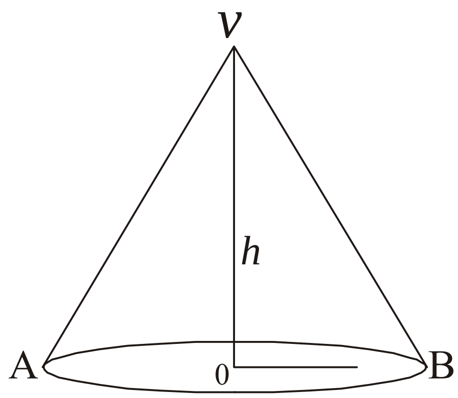
iv) Identify the maximum and minimum values of f out of the values calculated step 3. This maximum values will be the absolute maximum of f and the minimum valve will be the absolute minimum value of f.

ILLUSTRITATIVE EXAMPLES

1. Sand is pouring from a pipe at the rate of 12cm^3/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height in 4cm?

Solution :

Let 'r' be the radius 'h' be the height and v be the volume of the sand cone at any time 't'



then, $v = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(36h^2\right)h = 12\pi h^3$

$$\frac{dv}{dt} = 36\pi h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{3\pi h^2}$$

$$\frac{dh}{dt} at\left(h=4\right) = \frac{1}{48\pi} \text{cm / sec}$$

$$\left(\text{as } \frac{dv}{dt} = 12cm^2 / \sec\right)$$

Thus the height of the sand cone increasing as the rate of $\frac{1}{48\pi} cm / \sec$.

2. If the radius of a sphere is measured as 9cm with an error 0.03cm, then find the approximating error in calculating its volume.

Solution :

Let 'r' be the radius of a sphere and Δr be the error in measring the radius.

Then r = 9cm, Δr = 0.03cm

Let $V = \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = 4\pi r^2$ at r = 9 $\Rightarrow 4\pi \times 9^2 = 324\pi$

Let Δv be the error in v due to error Δr in r then,

$$\Delta v = \frac{dv}{dr} \cdot \Delta r = 324 \times \pi 0.03 = 9.72\pi cm^3$$

3. Show that the curves $x = y^2$ and $xy = K$ cut at right angles if $8K^2 = 1$

Solution :

The given curve are $x = y^2$ -----------(i) and $xy = K$ ---------------(ii)

From (i) and (ii) we obtain $y^3 = K \Rightarrow y = K^{2/3}$

So the two curves intersecs at the point $P\left(K^{2/3}, K^{1/3}\right)$ differentiating (i) *Wrt x* we get

$$I = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y} \Rightarrow m_1 = \left(\frac{dy}{dx}\right)P = \frac{1}{2K^{1/3}}$$

differentiating (ii) *wrt x* we get

$$1 \cdot y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_2 = \left(\frac{dy}{dx}\right)P$$

$$= \frac{-K^{1/3}}{K^{2/3}} = -\frac{1}{K^{1/3}}$$

For the curves (i) and (ii) to cut at right angles at P.

We must have $m_1 m_2 = -1$

$$\frac{-K^{1/3}}{K^{2/3}} \times \frac{-1}{K^{1/3}} = -1 \Rightarrow 2K^{2/3} = 1$$ cubeing on both sides we get $8K^2 = 1$

4. Find the intervals in which $f(x) = (x+1)^3 (x-3)^3$ is increasing or decreasing.

Solution :

We have,

$$f(x) = (x+1)^3 (x-3)^3$$

$$f'(x) = 3(x+1)^2 (x-3)^3 + (x+1)^3 \cdot (x-3)^2$$

$$f'(x) = 3(x+1)^2 (x-3)^3 (x+1+x-3)$$

For *f(x)* to be incresing we must have

$$= f'(x) > 0 \Rightarrow 6(x+1)^2 (x-1)(x-3)^2 > 0$$

$$\Rightarrow (x-1) > 0 \qquad \left(\because 6(x+1)^2 (x-3)^2 > 0\right)$$

$$x > 1$$

$$x \in (1, \infty)$$

$\therefore$ *f(x)* is increasing on $(1, \infty)$

For *f(x)* to be decreasing we must have $f'(x) < 0$

$$6(x+1)^2(x-3)^2(x-1) < 0$$

$$(x-1) < 0 \Rightarrow x < 1 \Rightarrow x \in (-\infty, 1)$$

$\therefore$ *f(x)* is decreasing on (–¥, 1)

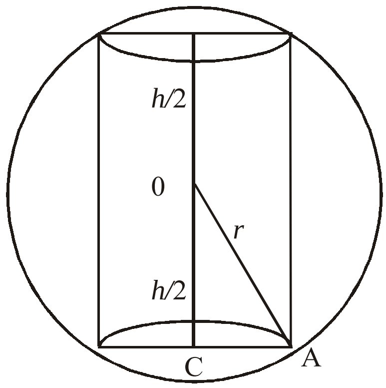
5. Find the volume of the larger cylinder that can be inscribed in a sphere of radius *r*cm.

Solution :

Let '*h*' be the height and R be the radius of the base of inscribed cylinder. Let V be the volume of the cylinder

Then, $V = \pi R^2 h$ ------------(i)

In ΔOCA we have



$$= r^2 = \left(\frac{h}{2}\right)^2 + R^2$$

$$\Rightarrow R^2 = r^2 - \frac{h^2}{4}$$

$$\therefore \; V = \pi\left(r^2 - \frac{h^2}{4}\right)h$$

$$= \pi r^2 h - \frac{\pi}{4}h^3$$

$$\frac{dv}{dh} = \pi r^2 - \frac{3\pi h^2}{4} \quad \text{and} \quad \frac{d^2v}{dh^2} = -\frac{3\pi h}{2}$$

For maximum or minimum values of V we must have

$$\frac{dv}{dh} = 0 \Rightarrow \pi r^2 - \frac{3\pi h}{4} = 0 \Rightarrow h^2 = \frac{4r^2}{3} \Rightarrow h = \frac{2}{\sqrt{3}}r$$

$$\frac{d^2v}{dh^2}\left(h = \frac{3r}{\sqrt{3}}\right) = \frac{-\pi r}{\sqrt{3}} < 0$$

Thus V is maximum when $h = \frac{2r}{\sqrt{3}}$

Put $h = \frac{2r}{\sqrt{3}}$ in $R^2 = r^2 - \frac{h^2}{4}$ we obtain $R^2 = \frac{\sqrt{2}}{3}r$

The maximum volume of the cylinder is given by

$$V = \pi R^2 h = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}$$

PROBLEMS FOR PRACTICE
GROUP–A (1 MARK QUESTION)

1. The radius of a spherical air bubble is increasing at the rate of 0.5cm/sec. At what rate is the volume of the bubble increasing when its radius is 1cm?
2. An edge of the variable cube is increasing at the rate of 50cm/sec. How fast is the volume of cube increasing when edge is 10cm long?
3. Find the slope of normal to the curve $y = \frac{1+\sin x}{\cos x}$ at $x = \frac{\pi}{4}$
4. Find the point on the curve $y = 2x^2 - 6x - 4$ at which the tangent is parallel to *x*–axis.
5. Find the equation of the tangent to the curve $\sqrt{x} + \sqrt{y} = a$ at the point $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$
6. Find the approximate value of $\sqrt{3}$.
7. If the radius of a circle is increased from 5cm to 5.1cm find the approximate increase in area.
8. Prove that the function $f(x) = x^3 + x^2 + x + 1$ does not have a maxima or minima.
9. Find the equation of the normal to the $x^{2/3} + y^{2/3} = 2$ at $(1, 1)$.

GROUP–B (4 MARKS QUESTIONS)

1. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).
2. Find the point on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are (i) Parallel to *x*–axis (ii) Parallel *y*–axis.
3. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent has equation $y = x - 1$.
4. Show that a closed right circular cylinder of given surface area and maximum volume is such that its height is equal to the diametre of the base.
5. Find the absolute maximum and absolute minimum value of the function $f(x) = 2\cos x + x, \ x \in (0, \pi)$.
6. It is gives that at $x = 1$ the function $f(x) = x^4 - 62x^2 + ax + 9$ attains its maximum value in the interval (0, 2). Find the value of a.
7. The total revenue received from the sale of *x* units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue at $x = 5$.
8. Find the intervals in which $f(x) = \sin x - \cos x$ $0 < x < 2\pi$ is increasing or decreasing.

9. Show that the function f is given $f(x) = \tan^{-1}(\sin x + \cos x)$ $x > 0$ in always increasing function in $\left(0, \frac{\pi}{4}\right)$.

10. A particle moves along the curve $y = \frac{\Delta}{3}x^3 + 5$. Find the points on the curve at which y–coordinate changes as fast as x–coordinate.

GROUP–C (6 MARKS QUESTIONS)

1. Show that the semi vertical angle of the right circular cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.
2. Show that the surface area of a closed curboid with a square base and given volume is minimum when it is a cube.
3. An open box with a square box is to be made out of a given quantity of metal sheet of area c^2. Show that the maximum volume of the box is $c^3/6\sqrt{3}$
4. Show that semivertical angle of right circular cone of given total surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$
5. Prove that the volume of the larger cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.
6. Show that the volume of the greater cylinder which can be inscribed in a cone of height 'h' and semivertical angle α is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.
7. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius 10cm is $\frac{20}{\sqrt{3}}cm$.
8. A grinder is in the form of a rectangle above which there is a semi circle. If the perametre of the window is p metres. Show that the window will allow the maximum possible light only when the radius of the semi circle is $\frac{p}{\pi + 4}cm$.
9. An open box with square base is to be made out of a given iron sheet of area 27 sq.cm. Show that the maximum volume of the box is 13.5cu.cm.
10. Given the sum of the perimetre of a square and a circle show that the sum of their areas is least when the side of the square is equal to the diametre of a circle.

ANSWERS
GROUP–A

1. $2\pi \, {cm^3}/{\sec}$
2. 1500 cm^3/sec.
3. $\left(\dfrac{-1}{2+\sqrt{2}}\right)$
4. $\left(\dfrac{3}{2}, \dfrac{-17}{2}\right)$
5. $2x + 2y = a^2$
6. 6.083
7. $\pi \cdot sq.cm$
8. —
9. $x - y = 0$

GROUP–B

1. $x + y - 3 = 0$
2. $(0, \pm 5), (\pm 2, 0)$
3. (2, –9)
4. —
5. max at $x = {\pi}/{6}$, is $\dfrac{\pi}{6} + \sqrt{3}$; min $\dfrac{5\pi}{6} - \sqrt{3}$ at $x = 5\dfrac{\pi}{6}$
6. 120
7. 66
8. $\left(0, {3\pi}/{4}\right), \left(\dfrac{7\pi}{4}, 2\pi\right)$
9. $\left(0, {\pi}/{4}\right)$
10. $\left(\dfrac{1}{2}, \dfrac{31}{6}\right)$ and $\left(-\dfrac{1}{2}, \dfrac{29}{6}\right)$

PART–II

INTEGRALS

SYNOPSIS :

1. If *f(x)* is derivative of a function *g(x)* then *g(x)* is known as antiderivative or integral of *f(x)*

   i.e. $\frac{d}{dx}(g(x)) = f(x) \Leftrightarrow \int f(x)dx = g(x)$

2. Derivative of a function is unique but the function can have infinite antiderivative or integrals.
3. $\int f(x)dx = g(x) + c$, where c is a constant of integration is known as indefinite integrals.
4. $\int dx = x$.
5. $\int c \cdot f(x)dx = c \cdot \int f(x)dx$
6. $\int \{f(x) \pm g(x)\}dx = \int f(x)dx = c \cdot f(x)dx$

SOME STANDARD INTEGRALS

7. $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$ n is a rational number.
8. $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$
9. In case of rational function if degree of numerator is equal to greater than degree of denominator then first we devide numerator by denominator and write it as

   $\frac{\text{Nr}}{\text{Dr}} = \text{Quotent} + \frac{\text{Remainder}}{\text{Denominator}}$ and then integrate.

10. $\int \sin x dx = -\cos x + c$
11. $\int \cos x dx = \sin x + c$
12. $\int \sec^2 x dx = \tan x + c$
13. $\int \mathrm{co\,sec}^2\, x dx = -\cot x + c$
14. $\int \sec x \tan x dx = \sec x + c$
15. $\int \mathrm{cos\,ec}x \cot x dx = -\mathrm{cos\,ec}x dx$
16. $\int \tan x dx = \log \sec x + c$ or $-\log \cos x + c$
17. $\int \cot x dx = \log \sin x + c$ or $-\log \mathrm{cosec} x + c$

18. $\int \sec x dx = \log\left|\sec x + \tan x\right| + c$

19. $\int \text{cosec} x dx = \log\left|\text{cosec} x - \cot x\right| + c$

20. $\int \frac{1}{x} dx = \log x + c$

21. $\int a^x dx = \frac{a^x}{\log a} + c$

22. $\int c^x dx = e^x + c$

23. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

24. $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

25. $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

26. $\int \frac{1}{\sqrt{a^2 + x^2}} dx = \log\left|x + \sqrt{a^2 + x^2}\right| + c$

27. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + c$

28. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a + x}{a - x}\right| + c$

29. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

30. $\int \sqrt{a^2 + x^2} dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \log\left|x + \sqrt{a^2 + x^2}\right| + c$

31. $\int \sqrt{x^2 - a^2} dx = \frac{x^2}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + c$

32. $\int f(x) \cdot g(x) dx = f(x) \int g(x) dx - \int \left(f'(x) \int g(x) dx \right) dx$ this is called integration log parts.

33. $\int e^x \left[f(x) + f'(x) \right] dx = e^x f(x) + c$

34. To make ax^2+bx+c as a perfect square, first note that coefficient of x^2 should be one and positive.

35. **Integration by partial fraction :**

First we must that the degree of numerator is less than degree of denominator, if not divide numerator and denominator and write it as —

$\frac{\text{Nr}}{\text{Dr}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$ and perceived for partial fractions of $\frac{\text{Remainder}}{\text{Denominator}}$.

i) When factors is denominator are linear and non repeated.

$\frac{P(x)}{(x+a)(x+b)(x+c)} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}$ where A, B, C are to be defeussed.\

ii) When factors in denominator are linear but repeated

$$\frac{P(x)}{(x-a)^2(x+b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x+b}$$

iii) When factors is denominator are quadratic, non repeated

$$\frac{P(x)}{(x+a)(x^2+b)} = \frac{A}{x+a} + \frac{Bx+c}{x^2+b}$$

iv) To evaluate $\int \frac{ax+b}{Px^2+qx+r}dx$

We write $ax+b = A\frac{d}{dx}(Px^2+qx+r)B$

Some Special Integrals :

i) $\int \frac{1}{(ax+b)\sqrt{cx+d}}dx$ put $\sqrt{cx+d} = t$

ii) $\int \frac{1}{a+b\sin x}dx$, $\int \frac{1}{a+b\cos x}dx$, $\int \frac{1}{a\cos x+b\sin x}dx$

In this case write $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

where $\tan x/2 = t$ and proceed to perfect square.

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS :

i) Let *f(x)* be a continous function on the closed interval [a, b] and let A(x) be the area function then $A'(x) = f(x)$ for all $x \in \{a, b\}$

ii) **Second Fundamental Theorem of Integral Calculus :**

Let f be a continous function on the closed interval (a, b) and $g(x)$ be the antiderivative of f then $\int_a^b f(x)dx = g(b) - g(a)$.

PROPERTIES OF DEFINITE INTEGRALS :

i) $\int_a^b f(x)dx = 0$

ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad a < c < b$

iv) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

vi) $\int_0^2 f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

vii) $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(x)\, f(2a-x) \quad = f(x)$

viii) $\int_0^{2a} f(x)dx = 0$ if $f(2a-x) = -f(x)$

ix) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even function $f(-x) = f(x)$

= 0 if $f(x)$ is odd function $f(-x) = -(x)$

INTEGRALS AS A LIMIT OF SUMS :

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left\{ f(x) + f(a+h) + f(a+2h) + ... + (a+(n-1)h)\right\}$$

where $h = \dfrac{b-a}{n}$

The following results are for evaluating questions based on limit of sums.

i) 1 + 2 + 3 + ... + (n – 1) = S(n – 1) = $\dfrac{n(n-1)}{2}$

ii) $1^2 + 2^2 + ... + (n-1)^2 = \Sigma(n-1)^2 = \dfrac{(n-1)n(2n-1)}{6}$

iii) $1^3 + 2^3 + ... + (n-1)^3 = \Sigma(n-1)^3 = \left[\dfrac{(n-1)n}{2}\right]^2$

iv) $a+ar+ar^2+\dots\dots\dots\dots+ar^{n-1}=\dfrac{a(r^n-1)}{r-1} \quad (r\neq 1)$

36. The value of definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integrators that we choose to represent the independent variable.
If the independent variable is denoted t or s instead of 'x'. We write —
$\int_a^b f(t)\,dt$ or $\int_a^b f(s)\,ds$ instead of $\int_a^b f(x)\,dx$. Hence variable of integration is called dummy variable.

ILLUSTRATIVE EXAMPLES

Emaple-1 : $\int \sin 3x \cos x dx = \frac{1}{2}\int 2\sin 3x \cos dx$

$$= \frac{1}{2}\int (\sin 8x - \sin 2x)\,dx$$

$$= \frac{1}{2}\left(-\frac{\cos 8x}{8}+\frac{\cos 2x}{2}\right)+c$$

Example-2 : $\int \frac{\sec x \tan x dx}{\sqrt{4-\sec^2 x}}$ put sec$x$ = t, sec$x$ tanxdx = dt

$$\int \frac{dt}{\sqrt{4-t^2}} = \sin^{-1}\left(\frac{t}{2}\right)$$

$$= \sin^{-1}\left(\frac{\sec x}{2}\right)+c$$

Example-3 : $\int \sqrt{\frac{a+x}{x}}dx$, put $x = at^2 \Rightarrow dx = 2atdt$

$$\int \sqrt{\frac{a+at^2}{at^2}} \times 2atdt = 2a\int \sqrt{1+t^2}\,dt$$

$$= 2a\left[\frac{t}{2}\sqrt{1+t^2}+\frac{1}{2}\log\left|t+\sqrt{1+t^2}\right|\right]+c$$

$$= 2a\left[\frac{1}{2}\sqrt{\frac{x}{a}}\cdot\sqrt{1+\frac{x}{a}}+\frac{1}{2}\log\left|\sqrt{\frac{x}{a}}+\sqrt{1+\frac{x}{a}}\right|\right]+c$$

$$= \sqrt{ax}\sqrt{1+\frac{x}{a}}+a\log\left|\sqrt{\frac{x}{a}}+\sqrt{1+\frac{x}{a}}\right|+c$$

Example-4 : Evaluate $\int e^{3x}\sin 2x dx$

Solution :

$$\mathrm{I} = \sin 2x \cdot \frac{e^{3x}}{3} - \int 2\cos 2x \cdot \frac{e^{3x}}{3}\, dx$$

$$= \frac{1}{3}e^{3x}\sin x - \frac{2}{3}\left[\cos 2x \cdot \frac{e^{3x}}{3} - \int (-2\sin 2x)\frac{e^{3x}}{3}\, dx\right]$$

$$\frac{13}{\not 9}\mathrm{I} = \frac{(3\sin 2x - 2\cos 2x)e^{3x}}{\not 9}$$

$$\mathrm{I} = \frac{e^{3x}}{13}(3\sin 2x - 2\cos 2x) + c$$

Example-5 : Evaluate $\int \frac{1}{\sqrt{(x-a)(x-b)}}\, dx$

Solution : $\int \frac{1}{\sqrt{(x-a)(x-b)}}\, dx = \int \frac{1}{\sqrt{x^2-(a-b)x+(a+b)}}dx$

$$= \int \frac{1}{\sqrt{\left(x-\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}}$$

$$= \log\left|\left(x-\frac{a+b}{2}\right) + \sqrt{(x-a)(x-b)}\right| + c$$

Example-6 : Evaluate $\int \frac{3x+2}{(x-1)(2x+3)}\, dx$

Solution : $\int \frac{3x+2}{(x-1)(2x+3)}\, dx = \frac{A}{x-1} + \frac{B}{2x+3}$

$3x+2 = A(2x+3) + B(x-1)$

on comparing $2A+B=3,\ \ 3A-B=2$

on solving for A and B we get A = 1, B = 1

Substitute and integrate we get

$$\int \frac{3x+2}{(x-1)(2x+3)}\, dx = \int \frac{1}{x-1}\, dx + \int \frac{1}{2x+3}\, dx$$

$$= \log|x-1| + \frac{1}{2}\log|2x+3| + c$$

Example-7 : $\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$

Solution : $\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$

Let $\tan x = t^2 \Rightarrow x^2 \tan^{-1}\left(t^2\right)$

$$dx = \frac{2t}{1+t^4}dt$$

$$\int\left(t+\frac{1}{t}\right)\ \frac{2t}{t^4+1}dt$$

$$=2\int\frac{t^2+1}{t^4+1}dt$$

$$=2\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt$$

$$=\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2} \qquad \text{Let } t-\frac{1}{t}=z$$

$$=2\int\frac{1}{z^2+2}d-z \qquad \left(1+\frac{1}{t^2}\right)dt = dz$$

$$=\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+c=\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)$$

$$=\sqrt{2}\tan^{-1}\left(\tan x-1\Big/\sqrt{2}\tan x\right)+c$$

Example-8 : Evaluate $\int_0^2(2x+3)dx$ as a limit of sums

$\int_0^2(2x+3)dx$, here $a = 0$, $b = 2$, $h=\frac{2-0}{n}\Rightarrow nh=2$, $f(x)=2x+3$

$$=\lim_{h\to o}h\left[f(x)+f(0+h)+(0+2h)+.............+t(0+n+2h)\right]$$

$$=\lim_{h\to 0}h\left[3+(2h+3)+(4h+3)+.................+2(n-1)h+3\right]$$

$$=\lim_{h\to 0}h\left[2h(1+2+3+...................+n-1)+3n\right]$$

$$= \lim_{h \to 0} h\left[2h \cdot \frac{n(n-1)}{2} + 3n\right] = \lim_{h \to 0} nh(nh-h) + (3nh)$$

$$= 2(2-h) + 6 = 4 + 6 = 10$$

Verification :

$$\int_0^2 (2x+3)\,dx = \left[\frac{2x^2}{2} + 3x\right]_0^2 = 4 + 6 = 10$$

Hence verified.

Example-9 : Using properties of definite integral evaluate $\int_0^{\pi/2} \frac{\sin^2 x}{\cos^2 x + \sin^2 x}$.

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\cos^2 x + \sin^2 x}\,dx \quad \text{---------------------- (i)}$$

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$= \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)dx}{\sin^2\left(\frac{\pi}{2} - x\right) + \cos^2\left(\frac{\pi}{2} - x\right)} = \int_0^{\pi/2} \frac{\cos^2 x dx}{\sin^2 + \cos^2 x}\,dx \quad \text{-------------- (ii)}$$

By adding (i) and (ii) we get —

$$2I = \int_0^{\pi/2} \frac{\cos^2 x + \sin^2 x}{\cos^2 x + \sin^2 x}\,dx = \int_0^{\pi/2} 1 \cdot dx = \pi/2$$

$$I = \pi/4$$

QUESTION FOR PRACTICE
GROUP–A (1 MARKS QUESTION)

1. Evaluate $\int \frac{5}{(7-3x)^3}\,dx$ Ans. : $\left(A : \frac{5}{6(7-3x)^2} + c\right)$

2. Evaluate $\int \tan^2 x dx$ Ans. : $(\tan x - x + c) : A$

3. Evaluate $\int 3^x e^x dx$ Ans. : $\frac{(3e)^x}{\log 3e} + c$

4. Evaluate $\int \frac{(1+\log x)^2}{x}$ Ans. : $\frac{1}{3}(1+\log x)^3 + c$

5. Evaluate $\int \log x dx$ Ans. : $(x\log x - x + c)$

6. Evaluate $\int x(4+x)^{1/4}\,\mathrm{dx}$ Ans. : $\left(\frac{4}{9}(4+x)^{9/4} - \frac{16}{5}(4+x)^{5/4} + c\right)$

7. Evaluate $\int \frac{1}{x\left(x^7+1\right)}\mathrm{dx}$ Ans. : $\left(\frac{1}{7}\log\left|1+\frac{1}{x^7}\right| + c\right)$

8. Evaluate $\int \sec x\cdot(\sec x + \tan x)\,\mathrm{dx}$ Ans. : $\left[\log(\sec x + \tan x)^2 + c\right]$

9. Evaluate $\int \frac{1}{\sqrt{(2-x)^2-3}}dx$ Ans. : $-\log\left|2-x+\sqrt{(2-x)^2-3}\right|$+c

10. Evaluate $\int e^x(\sin x + \cos x)dx$ Ans. : $\left[e^x\cdot\sin x + c\right]$

GROUP–B (4 MARKS)

Evaluate the following integrals :—

1. $\int \frac{1}{\sin x(2+\cos x)}dx$ Ans.: $\left(\frac{1}{6}\log|1-\cos x| - \frac{1}{2}\log|1+\cos x| + \frac{1}{3}\log|2+\cos x| + c\right)$

2. $\int \frac{11x+8}{x^2(3x+8)}dx$ Ans.: $\left(\log|x| - \frac{1}{x} - \log|3x+8| + c\right)$

3. $\int \frac{1}{(x+1)\sqrt{x^2-1}}dx$ Ans.: $\left(\sqrt{1-\frac{2}{x+1}} + c\right)$

4. $\int_0^{\pi/4}\sqrt{1+\sin 2x}dx$ Ans.: (1)

5. $\int \frac{dx}{1+x^4}$ Ans.: $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{2\sqrt{x}}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2x}+1}{x^2+\sqrt{2x}+1}\right| + c$

6. $\int x\cos^{-1}x dx$ Ans.: $\frac{x^2}{2}\cos^{-1}x + \frac{1}{4}\left[\sin^{-1}x - x\sqrt{1-x^2}\right] + c$

7. $\int \frac{1}{x^3+x^2+x+1}dx$ Ans.: $\frac{1}{4}\log\left(x^2+1\right) + \frac{1}{2}\tan^{-1}x - \frac{1}{2}\log|x+1| + c$

8. $\int \sqrt{\cot x dx}$ Ans.: $\frac{-1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot x - 1}{\sqrt{2\cot x}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{\cot x - \sqrt{2\cos x} + 1}{\cos x + \sqrt{2\cos x} + 1}\right|$

9. $\int\left\{\log(\cot x)+\frac{1}{(\cot x)^2}\right\}dx$ Ans.: $x\log(\cot x)-\frac{x}{\cot x}+c$

10. $\int\frac{\tan x\cdot\sec^2 x}{1-\tan^2 x}dx$ Ans.: $\frac{-1}{2}\log\left(1-\tan^2 x\right)+c$

11. $\int e^x \operatorname{cosec}x(1-\cot x)dx$ Ans.: $e^x\operatorname{cosec}+c$

12. $\int\frac{xe^{2x}}{(1+2x)^2}$ Ans.: $\frac{1}{4}\frac{e^{2x}}{1+2x}+c$

13. $\int\sqrt{x^2+8x+4}dx$ Ans.: $\frac{x+4}{2}\sqrt{x^2+8x+4}-6\log(x+4)+\sqrt{x^2+8x+4+c}$

14. $\int\frac{x^2\tan^{-1}x}{1+x^2}dx$ Ans.: $x\tan^{-1}x-\frac{1}{2}\log\left(1+x^2\right)-\frac{1}{2}\left(\tan^{-1}x\right)^2+c$

15. $\int\frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$ Ans.: $\frac{2}{3(a-b)}\left[(x+a)^{3/2}-(x+b)^{3/2}\right]+c$

16. $\int\operatorname{cosec}x\log(\operatorname{cosec}x-\cot x)dx$ Ans.: $\frac{\left[\log(\operatorname{cosec}x-\cot x)\right]^2}{2}+c$

17. $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$ Ans.: $\frac{x}{2}\cos^{-1}x-\frac{1}{2}\sqrt{1-x^2}+c$

18. $\int\sqrt{\frac{a-x}{a+x}}dx$ Ans.: $-a\cos^{-1}\left(\frac{x}{a}\right)+\sqrt{a^2-x^2}+c$

19. $\int\left(e^{x\log h}+e^{a\log x}+e^{a\log h}\right)dx$ Ans.: $\frac{a^x}{\log a}+\frac{x^{a+1}}{a+1}+a^a\cdot x+c$

20. $\int\sqrt{\frac{\sin(x-a)}{\sin(x+a)}}dx$ Ans.: $-\cos a\cdot\sin^{-1}\left(\frac{\cos x}{\cos a}\right)-\sin a\log\left|\sin x+\sqrt{\sin^2 x-\sin^2 a}\right|$

21. $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$ Ans.: $-e^x\cot\frac{x}{2}+c$

22. $\int e^x\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)dx$ Ans.: $e^x\tan x+c$

23. $\int\frac{\sin x\cos x dx}{\sin^2+\cos^2 x}$ Ans.: $\frac{1}{2}\tan^{-1}\left(\tan^2 x\right)+c$

24. $\int \frac{\sin x+\cos x}{\sin^2 x+\cos^2 x}dx$ Ans.: $\tan^{-1}(\sin x-\cos x)+\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}\right|+c$

25. $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx$ Ans.: $x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\frac{x}{2}+c$

26. $\int \frac{1}{\cos^2 x(1-\tan x)^2}dx$ Ans.: $\frac{1}{1-\tan x}+c$

27. $\int \frac{x^3+x}{x^4-9}dx$ Ans.: $\frac{1}{x}\log\left|x^4-9\right|+\frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right|+c$

28. $\int x\log(x+1)dx$ Ans.: $\frac{x^2}{2}\log|x+1|-\frac{1}{4}(x-1)^2-\frac{1}{2}\log|x+1|+c$

29. $\int \frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\cos x\right]}{x^4}dx$ Ans.: $\left(\because \frac{-1}{3}\left(1+\frac{1}{x^2}\right)\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+c\right)$

30. $\int \frac{1}{\cos(x+a)\cos(x+b)}dx$ Ans.: $\frac{1}{\sin(a-b)}\{\log\sec(x+a)-\log(\sec x+b)\}+c$

GROUP–C (6 MARK QUESTIONS)

1. $\int_0^{\pi/2}\frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}d\theta$ Ans.: $\left(\frac{\pi}{2}\right)$

2. $\int_{-1}^{2}|x+1|+|x|+|x-1|dx$ Ans.: $\left(\frac{19}{2}\right)$

3. $\int_0^{\pi}\frac{x}{a^2\cos^2 x+b^2\sin^2 x}dx$ Ans.: $\left(\frac{\pi^2}{2ab}\right)$

4. $\int_0^{\pi/2}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$ Ans.: $\left(\pi\sqrt{2}\right)$

5. Prove that $\int_0^a f(x)dx=\int_0^a f(a-x)dx$ and hence prove that $\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx=\pi/4$

6. $\int_0^{\pi/2}\log\sin x dx$ Ans.: $\left(-\frac{\pi}{2}\log 2\right)$

7. Evaluate the following limit of sum.

   a) $\int_0^3(2x^2-5)dx$ Ans.: 3

b) $\int_1^4 \left(x^2 - x\right) dx$ Ans.: $\frac{27}{2}$

c) $\int_0^2 \left(x^2 + x + 1\right) dx$ Ans.: $\frac{20}{3}$

d) $\int_0^3 \left(2x^2 + 3x + 5\right) dx$ Ans.: $\frac{118}{3}$

e) $\int_{-1}^1 e^x dx$ Ans.: $e - \frac{1}{e}$

f) $\int_0^4 \left(x + e^{2x}\right) dx$ Ans.: $\frac{15 + e^8}{2}$

8. $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$ Ans.: $\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2\sqrt{x - x^2}}{\pi} - x + c$

9. $\int_0^{\pi/4} \left(\frac{\sin x + \cos x}{9 + 16\sin 2x}\right) dx$ Ans.: $\frac{1}{40}\log 9$

10. $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ Ans.: $\frac{\pi}{2}(\pi - 2)$

11. $\int_{-1}^{3/2} \left|x \sin \pi x\right| dx$ Ans.: $\frac{3}{\pi} + \frac{1}{\pi 2}$

12. $\int \frac{\sin 2x \cdot \cos 2x dx}{\sqrt{9 - \cos^2(2x)}}$ Ans.: $-\frac{1}{4}\sin^{-1}\left[\frac{1}{3}\cos^2 2x\right] + c$

13. $\int \frac{x^4 dx}{(x-1)\left(x^2+1\right)} dx$ Ans.: $\frac{x^2}{2} + x + \frac{1}{2}\log\left|x-1\right| - \frac{1}{4}\log\left|x^2+1\right| - \frac{1}{2}\tan^{-1} x + c$

14. $\int_0^{\pi} \log(1 + \cos x) dx$ Ans.: $-\pi \log 2$

15. $\int e^{ax} \sin bx dx$ Ans.: $\frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$

ANSWERS

1. $\frac{5}{6(7-3x)^2} + c$

2. $\tan x - x + c$

3. $\frac{(3e)^x}{\log 3e}+c$

4. $\frac{1}{3}(1+\log x)^3+c$

5. $x\log x-x+c$

6. $\frac{4}{9}(4+x)^{9/4}-\frac{11}{5}(4+x)^{5/4}+c$

7. $\frac{1}{7}\log\left|1+\frac{1}{x^7}\right|+c$

8. $\log(\sec x+\tan x)^2+c$

9. $-\log\left|2-x+\sqrt{(2-x)^2-3}\right|+c$

10. $e^x \sin x+c$

FOR GROUP–B

1. $\frac{1}{6}\log|1-\cos x|-\frac{1}{2}|1+\cos x|+\frac{1}{3}\log|2+\cos x|+c$

2. $\log|x|-\frac{1}{x}-\log|3x+8|+c$

3. $\sqrt{1-\frac{2}{x+1}}+c$

4. 1

5. $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{2\sqrt{2}}\right)-\frac{1}{2}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|+c$

6. $\frac{x^2}{2}\cos^{-1}x+\frac{1}{4}\left(\sin^{-1}x-x\sqrt{1-x^2}\right)+c$

7. $\frac{1}{4}\log\left(x^2+1\right)+\frac{1}{2}\tan^{-1}x-\frac{1}{2}\log|x+1|+c$

8. $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot x-1}{\sqrt{2\cot x}}\right)-\frac{1}{2\sqrt{2}}\log\left|\frac{\cot x-\sqrt{2\cot x}+1}{\cot x+\sqrt{2\cot x}+1}\right|+c$

9. $x\log(\cot x)-\frac{x}{\cot x}+c$

10. $-\frac{1}{2}\log\left(1-\tan^2 x\right)+c$

11. $e^x \text{cosec} x + c$

12. $\frac{e^{2x}}{4(1+2x)}+c$

13. $\frac{x+4}{2}\sqrt{x^2+8x+4}-6\log(x+4)+\sqrt{x^2 8x+4}+c$

14. $x\tan^{-1}x-\frac{1}{2}\log\left(1+x^2\right)-\frac{1}{2}\left(\tan^{-1}x\right)^2+c$

15. $\frac{2}{3(a-b)}\left[(x+a)^{3/2}-(x+b)^{3/2}\right]+c$

16. $\left[\frac{\log(\text{cosec}x-\cot x)}{2}\right]^2+c$

17. $\frac{x}{2}\cos^{-1}x-\frac{1}{2}\sqrt{1-x^2}+c$

18. $-a\cos^{-1}\left(\frac{x}{a}\right)+\sqrt{a^2-x^2}+c$

19. $\frac{a^x}{\log a}+\frac{x^{a+1}}{a+1}+a^a\cdot x+c$

20. $-\cos a\cdot\sin^{-1}\left(\frac{\cos x}{\cos a}\right)-\sin a\cdot\log\left|\sin x+\sqrt{\sin^2 x-\sin^2 a}\right|+c$

21. $-e^x\cot x/2+c$

22. $e^x\tan x+c$

23. $\frac{1}{2}\tan^{-1}\left(\tan^2 x\right)+c$

24. $\tan^{-1}(\sin x-\cos x)+\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}\right|+c$

25. $x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\frac{x}{2}+c$

26. $\frac{1}{1-\tan x}+c$

27. $\frac{1}{4}\log\left|x^4-9\right|+\frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right|+c$

28. $\frac{x^2}{2}\log|x+1|-\frac{1}{4}(x-1)^2-\frac{1}{2}\log|x+1|+c$

29. $\frac{-1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+c$

30. $\frac{1}{\sin(a-b)}\left\{\log\sin(x+a)-\log\left(\sin(x+b)\right)\right\}$

FOR GROUP–C

1. $\pi/2$
2. $\left(\frac{19}{2}\right)$
3. $\frac{\pi^2}{2ab}$
4. $\pi\sqrt{2}$
5. —
6. $\frac{-\pi}{2}\log 2$
7. a) 3

   b) $\frac{27}{2}$

   c) $\frac{20}{3}$

   d) $\frac{118}{3}$

   e) $e-\frac{1}{e}$

   f) $\frac{15+e^8}{2}$

8. $\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x}+\frac{2\sqrt{x-x^2}}{\pi}-x+c$

9. $\frac{1}{40}\log 9$

10. $\frac{\pi}{2}(\pi-2)$

11. $\frac{3}{\pi}+\frac{1}{\pi 2}$

12. $-\frac{1}{4}\sin^{-1}\left(\frac{1}{3}\cos^2 2x\right)+c$

13. $\frac{x^2}{2}+x+\frac{1}{2}\log|x-1|-\frac{1}{4}\log|x^2+1|-\frac{1}{2}\tan^{-1}x+c$

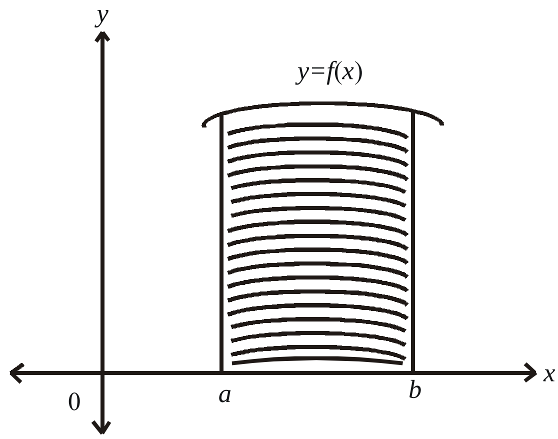
14. $-\pi\log 2$

15. $\frac{e^{ax}}{a^2+x^2}(a\sin bx-b\cos bx)+c$

APPLICATIONS OF INTEGRATION

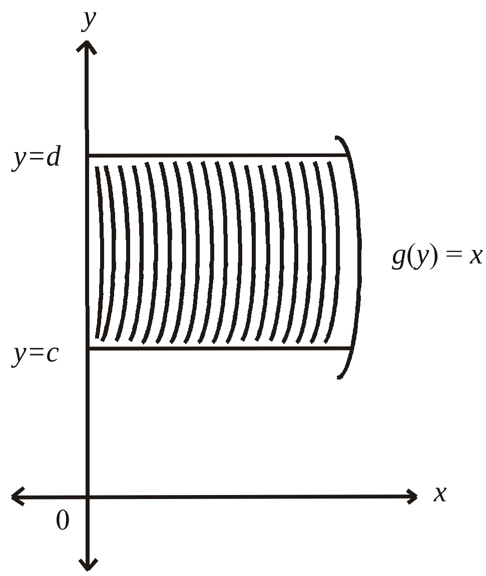
SYNOPSIS :

1. Area bounded by the curve $y = f(x)$, the x–axis and between the ordinates at $x = a$ and $x = b$ is given by $\int_a^b y\,dx = \int_a^b f(x)\,dx$.



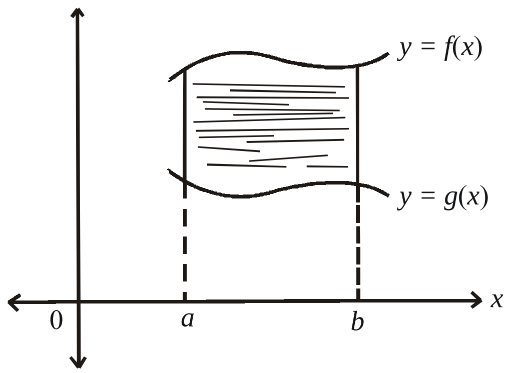
2. Area bounded by the curve $y = f(x)$, the y–axis and between absussa at $y = c$ and $y = d$ is given by

$$\text{Area} = \int_c^d x\,dy = \int_c^d g(y)\,dy$$



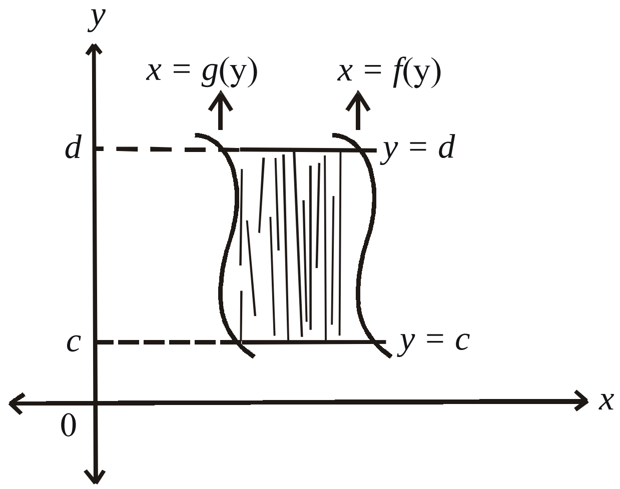
3. If the areas lies below x–axis or left side of y–axis then it is negative and in such a case we take its absolute value (numerical value).
4. Area bounded by two curves $y = f(x)$ and $y = g(x)$ such that $0 \le g(x) \le f(x)$ for all

$x \in [a,b]$ and between the ordinate at $x = a, x = b$ is given by $\int_a^b \{f(x) - g(x)\} dx$.



5. Area bounded by the curves $x = f(y)$ and $x = g(y)$ such that $0 \le g(y) \le f(y)$ and between the abscissa at $y = c$ and $y = d$ is given by—

$$\text{Area} = \int_c^d \{f(y) - g(y)\} dy$$



6. If $f(x) \ge g(x)$ in [a,c] and $f(x) \le g(x)$ in [c,d] where $a < c < b$ then area of the region bornded by the curves is given as Area $\int_a^c \{f(x) - g(x)\} dx + \int_c^b \{g(x) - f(x)\} dx$

ILLUSTRATIVE EXAMPLES

Example-1 : Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

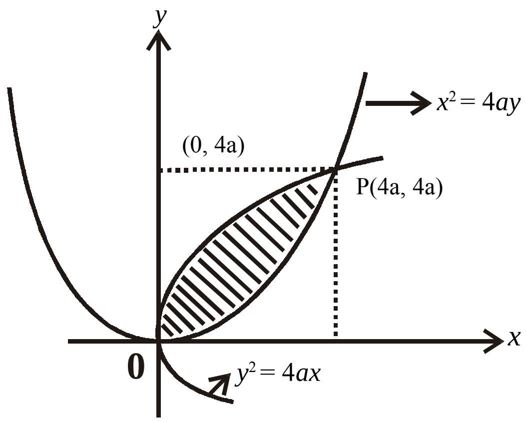
Proof : Parabolas are $y^2 = 4ax$ symmetrical to *x*–axis and $x^2 = 4ay$ symmetrical to *y*–axis.

on solving (1) and (2) we get

$$\left(\frac{x^2}{4a}\right) = 4ax$$

$$x^4 - 64a^3x = 0$$

$$x = 0,\ x = 4a$$



A_1 : curve $y^2 = 4ax$, x–axis, $x = 0$ and $x = 4a$

$$A_1 = \int_0^{4a} \sqrt{4ax}\,dx = 2\sqrt{a}\cdot\left[\frac{2}{3}x^{3/2}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}(4a)^{3/2} = \frac{32a^2}{3}$$

A_2 : curve $x^2 = \Delta ay$ x–axis and between x–axis $x = 0$, $x = 4a$

$$A_2 = \int_0^{4a} \frac{x^2}{4a}dx = \left[\frac{1}{12a}x^3\right]_0^{4a} = \frac{16a^2}{3}$$

Area enclosed $= A_1 - A_2 = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}\,sq.$ unit .

Example-2 : Find the area of the region $\{(x,y): y^2 \le 4x \;\; 4x^2 + 4y^2 \le 9\}$

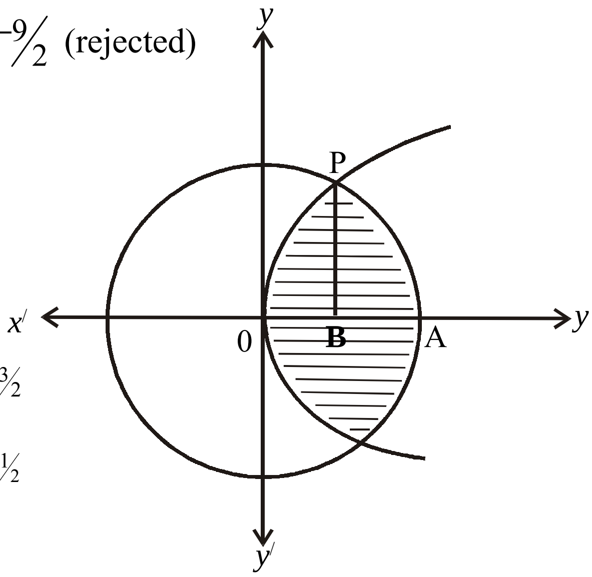
Solution : Curves are $y^2 = 4x$ parabola

symmetric to x–axis and $x^2 + y^2 = \frac{9}{4}$ circle with centre (0, 0) and radius $= 3/2$

both the curves are symmetric to x-axis as both the functions are even with respect to y

on solving equation (1) and (2) we get $x = 1/2,\ -9/2$ (rejected)

$$\therefore B\left(\frac{1}{2},0\right),\ A\left(\frac{3}{2},0\right)$$



Area of the region

$$= 2\left[\int_0^{1/2} 2\sqrt{x}\,dx + \int_{1/2}^{3/2}\sqrt{\frac{9}{4} - x^2}\,dx\right]$$

$$= \left[\frac{4}{3}\left(x^{3/2}\right)\right]_0^{1/2} + \left[\cancel{2}\frac{x}{\cancel{2}}\sqrt{\frac{9}{4} - x^2} + \cancel{2}\cdot\frac{9}{\cancel{8}_4}\sin^{-1}\frac{2x}{3}\right]_{1/2}^{3/2}$$

$$= 2\frac{\sqrt{2}}{3} + \frac{9\pi}{8} - \frac{\sqrt{2}}{2} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$= \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{6}$ square units.

Example-3 : Find the area of the region bounded by the lines $y = 2x+1$, $y = 3x+1$, $x = 4$ using integration.

Solution : Consider $y = 2x+1$ and $y = 3x+1$

$x = 0 \Rightarrow y = 1$ $\quad$ $x = 0,\ y = 1$

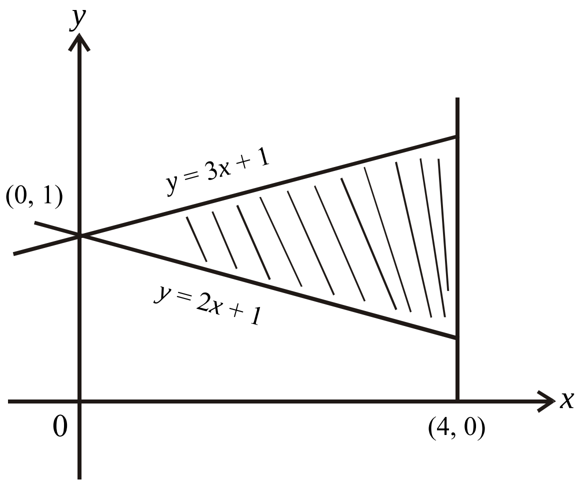
$x = 1,\ y = 3$ $\quad$ $x = 1,\ y = 4$

Area enclosed $\int_0^4 (y, -y)\,dx$

Where, $y_1 : 3x+1,\ y_2 = 2x+1$

$= \int_0^4 (3x+1-2x-1)dx$

$= \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = 8$ square units.



QUESTIONS FOR PRACTICE
GROUP–B (4 MARKS)

1. Find the area of the region bounded by two parabolas $y^2 = 4x$ and $x^2 = 4y$.
2. Find the area bounded by lines $x + 2y = 2,\ y - x = 1$ and $2x + y = 7$.
3. Sketch the region common to the circle $x^2 + y^2 = 16$ and $x^2 = 6y$. Also find the area of the region using integration.
4. Area of the region bounded by $y = 9x^2$ and $x = 0,\ y = 1$ and $y = 4$ and the first quadrant.
5. Find the area of the region bounded by $x^2 + y^2 = a^2$.
6. Find the area of the region bounded $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
7. Find the area of the triangle DABC formed by A(2, 3), B(4, 7), C(6, 2).
8. Find the area of the region $y^2 = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.
9. Find the area of the region bounded by the circles $x^2 + y^2 = 16$ and the line $y = x$ in the first quadrant.
10. Find the area of the region formed by the curve $x^2 = 4y$ and the line $x = 4y - 2$

GROUP–C (6 MARKS)

1. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$, and the straight line $\frac{x}{a}+\frac{y}{b}=1$
2. Find the area of the region enclosed between the two circles $x^2+y^2=1$ and $(x-1)^2+y^2=1$.
3. Using integration find the areas of the region given below
   $\{(x,y):0\le y\le x^2+1,0\le y\le x+1,0\le x\le 2\}$
4. Find the area of the following region
   $\{(x,y):y^2\le 4x,4x^2+4y^2\le 9\}$
5. Find the area of the region given by $\{(x,y):x^2\le y\le |x|\}$
6. Find the area bounded by $y=6x-x^2$ and $y=x^2-2x$.
7. Find the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the ordinates $x=ae$ and $x=0$ Where $b^2=a^2(1-e^2)$ and $e<1$.
8. Draw the raugh sketch $y=\sin 2x$ and determine the area enclosed by the curve, the x–axis and the lines $x={}^{\pi}\!/_{4}$ and $x={}^{3\pi}\!/_{4}$.
9. Make a rough sketch of the region given below and find the area using integration.
   $\{(x,y):0\le y\le x^2+3,0\le y\le 2x-3,0\le x\le 3\}$
10. Using the method of integration find the area bounded by the curve $|x|+|y|=1$.

ANSWERS

GROUP–B

1. $\frac{16}{3}$ sq.units
2. 6 sq.units
3. $\frac{23}{6}$ sq.units
4. $\frac{14}{9}$ sq.units

5. πa^2
6. πab sq.units
7. 9 sq.units
8. $\frac{21}{2}$ sq.units
9. 2p sq.units
10. $\frac{9}{8}$ sq.units

GROUP–C

1. $\frac{ab}{4}(\pi - 2)$ sq.units
2. $\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq.units
3. $\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$ sq.units
4. $\frac{23}{06}$ sq.units
5. $\frac{1}{3}$ sq.units
6. $\frac{64}{3}$ sq.units
7. $ab\left[e\sqrt{1-e^2} + \sin^{-1} e\right]$ sq.units
8. 1 sq.units
9. $\frac{50}{3}$ sq.units
10. 2 sq.units